ROBUSTLY OPTIMAL WIND TURBINE WITH SWIRL EXPANSION AND DECAY

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Abstract- *The application of propeller blade element 'BEM' and especially the refined 'SET' general momentum theories to windmills is self-contradictory. The windmill optimal interferences aren't small as the BEM assumes. 'SET' makes the windspeed drop at the rotor a fixed fraction < ½ of that downstream, whereas the ring vorticity of its wake induces below ½ more towards the hub. Much worse, the SET useful power rises to exceed the Betz limit at small ratio of tip to wind speeds, contrary to all reputable data and the BEM; and its large collateral wasted wake swirl power makes its gross power far exceed the undisturbed wind energy flux. Its wake suction below ambient edge pressure must vanish as the swirl dissipates; but then its inner flow would reverse.*

Instead the ***average*** *pressure in the expanded swirling wake must be near ambient, so there is negligible further net expansion when the swirl dissipates. The BEM induced fraction of ½ implies the waste of all the rotational energy added at the rotor; but if the swirl expands with pressure regain before dissipation, the power can be 3.5% higher and the fraction does fall below ½ towards the hub. This 'DESH' rotor is not soluble at small speed ratio, but at .57 matches a BEM core likely by bursting of the unstable hub vortex. A simple direct experimental verification is proposed further to good agreement with existing measurements and CFD. The design blade chord and pitch that make the new DESH optimum robust to small changes in the wind are compared to the analytic BEM robust values.*

Nomenclature:

a' the azimuthal interference factor
a half the net axial velocity change
A the axial interference at the rotor
B the number of blades
c the local blade chord
c_P local elemental power coefficient per strip of swept area
c_T local elemental thrust coefficient = Thrust/ $1/2\rho V^2$/ unit swept area
C_L lift coefficient $L/2\rho V^2$/ unit blade area
C_P rotor coefficient of power / undisturbed kinetic energy flux through the swept area

T- the rotor downwind thrust
e the streamtube area at the rotor vs its area at station 1
f the ratio of azimuthal induced flow to apparent wind at the rotor *v/W*
k reduced frequency $\Omega c/V$
P the dimensional power per unit length of span
p the absolute pressure
p_1 non-dimensional gauge pressure at station 1 $(p_1-p_0)/ \frac{1}{2}\rho V^2$
P_1 tip value of "
Q streamtube volume flux
*V*windspeed
$t = 1-A$ non dimensional axial velocity at rotor
W speed or magnitude of the apparent wind
u axial velocity at the rotor, behind with subscript
v azimuthal velocity behind the rotor
x local speed ratio $\Omega r/V$
X the tip speed ratio
y square of *x*
r unit radial vector
I the induction, half the velocity change the blades produce downstream in their wake
L airfoil lift vector
V True undisturbed wind vector
W The net apparent wind
α - ¾ chord angle of attack to **W**
α_{stall} the (dynamic) stall onset value of α
η - Glauert thrust efficiency c_T/c_P
φ - The true or complete apparent wind **W** angle to the blade path
λ - the blade pitch or angle of the 3/4 blade chord to the blade path
κ - $a'x^2$ circulation normalized by $2\pi T^2/\Omega$.
ρ fluid density
σ - true local solidity = blade chords/circumference of blade travel $Bc/ 2\pi r$
$\varpi = \pi\sigma/2 = Bc/4r$ half the net blade chord divided by the diameter
Ω- angular velocity of rotation in radians per unit time
subscript 0 denotes value in wake after swirl has dissipated station '0'
1 denotes value in wake at end of rotortube expansion station '1'

Keywords: Hawt; BEM; General Momentum Theory; wake expansion; vortex burst

Introduction –Wake Problems in Blade Element BEM & General Momentum (SET) Theories

This paper seeks to address the salient wake contradictions arising from the application of propeller blade element and especially general momentum theories to 'Hawts' horizontal axis wind turbines. In standard 'BEM', Blade Element Momentum, propeller theory, the vortex argument for the axial induced flow at the rotor being half that in the far wake assumes these changes to the wind V to be small so the wake streamtubes are cylinders. This is a valid approximation for the small rotor axial interference "*a*" of propellers of high efficiency. However using the same induced fraction to optimise the wind rotor gives *a* approaching ⅓ for 100% area expansion from the rotor or 200% from upwind, begging at least a correction of the wake induced fraction of ½ .

To calculate properly the self-induced flows requires an enormous computational effort. Joukowski's 'general' momentum theory offers a simpler, if indirect, way of estimating the effects of the expansion. For constant circulation along the blades, Glauert [1] used it to correct the propeller induced fraction to slightly more than ½. It considers not only the contracted slipstream but also its suction as its swirl squared, so the 'general momentum theory' might be more specifically named "Suction Expansion Theory" (SET). The suction decreases the thrust T and the higher induced flow raises the power P, both lowering the propeller efficiency $\eta=VT/P$ slightly. Glauert showed neglecting such and all terms in swirl squared, small for a propeller, reduced the SET to the standard (Blade Element) Momentum (BEM) theory.

He then used these simpler BEM equations with their induced fraction of ½ to numerically optimise the Hawt P (since solved analytically [2]), but the optimum swirl is not small. Versus a propeller, the suction and expansion SET effects are in fact reversed to be so positive as to even overcome the experimentally established and BEM-predicted loss of P at small ratio x of tip to wind speeds and exceed the Betz limit of 16/27 the undisturbed wind energy flux. [3]

The preoccupation of Joukowski and Glauert was propellers and Glauert died suddenly in 1934 before his manuscript was completely finalised, so he may well not have had time to fully appreciate, let alone resolve, the glaring windmill contradictions in the two theories. Here the expansion and suction effects will be separated and analysed to find the long-standing Hawt flaw in SET, and to better understand and refine the Hawt BEM.

Wake Expansion from the Vorticity and Stability Viewpoints

The vortex wake can be decomposed into trailing loops of vorticity, axially from the hub, bound in a blade, and spiralling downstream from the blade trailing edge. Those trailing spirals can be further broken down into axial and azimuthal segments. The axial segments match the opposed hub vortex in length back to the starting vortex for no net axial vorticity; and in the actuator disc continuum approximation, the azimuthal segments form trailing tubes of vortex rings.

The axial slowdown in the wake stretches these ring vortices, which doesn't change their circulation; but its packing them closer together does increase their infinite tube axial induction. The induced flow back at the rotor will be changed less from the unexpanded tube value because it is more determined by the weaker ring vortex sheet strength in the expansion zone. Thus the average induced flow fraction is expected to be less than ½. However just inside the outer edge of the rotor, the smaller jump across the weaker tip ring vortex tube is opposed by the smaller outside velocity due to induction against the wind by the wider far wake tube's projection beyond the tip. They must nearly cancel as computations [4] of wake expansion for tip vortex rings rise to an induced fraction ½ at the edge of the actuator disc, whereas SET predicts a radially uniform fraction < ½ for the same constant circulation. [3] However for a propeller, there can be little wake influence at the rotor tip outside the extension of the contracted wake, so the induced fraction there is > ½ , as SET predicts.

Deceleration often causes drastic flow changes like boundary layer separation and hydraulic jumps, so windmill expansion may not indeed be a reverse of propeller slipstream contraction, particularly with the heavy Hawt swirl at the hub, greatest when the interference rises above optimal for a non-robust design[2]. Flows with swirls theoretically develop reverse flows and multiple solutions in deceleration [5] roughly when their swirl speed exceeds their axial, the experimental criterion for a vortex to burst [6]. Equating these speeds behind the BEM optimal Hawt $2xa'=1-a$ or $\tan \frac{1}{2}\varphi = \frac{1}{2}$ at x=.18 and in its wake, $2xa'=1-2a$ or $\sin \varphi = \frac{1}{2}$ at x=1 , where xa' is v/V, v the swirl velocity, φ the angle of the apparent wind [2]. Whereas for Glauert's [1,p198] $\eta=\frac{3}{4}$ propeller with the wake kinetic energy flux the same ⅓ of the thrust power as the ideal windmill, $1+a \geq 1 >> 2xa' \rightarrow 0$ as $x \rightarrow 0$ so the mild swirl is stable and being safely contracted.

Very detailed calculations [7] for constant circulation windmill blades confirm the blade root vortices become unstable as they diffuse into a single continuous actuator axial vortex. In wind tunnel measurements on a Rutland rotor reduced to 3 blades, Masouh [8] saw merging of the root and tip vorticity within 1 diameter downstream, which probably indicates a massive vortex burst behind its bluff hub.

SET Angular Momentum, Head, and Pressure Equations

Just in front of an actuator disk rotating at angular velocity Ω, irrotationality requires the angular velocity induced by any axisymmetric shape of vortex wake to exactly cancel that induced by the bound vorticity of the actuator disc. Since this bound vortex sheet induces equally and oppositely on its two sides, if the (average) angular velocity increase across the disc is ω, the angular velocity induced by the wake alone on the disc must be $\frac{1}{2}\omega$ [1].

The ω increase is reacting to the torque Γ exerted by the flow on the blades: $d\Gamma = r^2 \omega \rho dQ$ (1) where ρ is the fluid density and the infinitesimal volume flux is $dQ=udS$, u the axial velocity at the rotor, and dS the annular area at radius r. The drop in flow total energy $\Delta H dQ = dP = \Omega d\Gamma$ is the power removed, so $\Delta H = \rho\, \Omega \omega r^2$ (2)

Since swirl kinetic energy is added across the disc, the pressure drop is $\Delta p = \rho(\Omega + \frac{1}{2}\omega)\omega r^2$ (3)

Now differencing the radial component of the full axisymmetric Navier Stokes equations across the disc gives $d\Delta p/dr = -\rho\omega^2 r$ a centrifugal balance even with radial velocity (axisymmetric and continuous across the disc). Substituting in (3) then shows $\omega \propto r^{-2}$ so the circulation is constant and then the radial vortex lines bend downstream at the axis into a single 'potential' vortex. Thus one might argue the BEM should have been optimised for a given tipspeed X subject to this constraint which will inevitably lower its best $Cp(X)$. Certainly this is the appropriate (and even more glaring) comparison for the known constant circulation SET solution.

Instead Glauert optimised the BEM independently at each r, which gave constant circulation behind the outer disc but constant swirl, not circulation, behind the root for inadequate suction $p-p_0=-5/16\, \rho V^2/2$ or a radial imbalance there in favour of an expansive jump somehow. It may be argued that given the discrete blade rather than axisymmetric porous nature of a real rotor, (1)-(3) are actual azimuthal averages and that square of the average swirl is not the average of swirl squared and can only be used some distance downstream where diffusion has smoothed out the swirl azimuthally. Then the BEM (and DESH) escape because they don't explicitly use the non-linear (3), but without (3) SET wouldn't close as follows..

If a wake annulus expands **ideally** from radius r to r_1 (Figure 1) the axial velocity drops to u_1

such that $u_1 \mathrm{d}r_1^2 = u\,\mathrm{d}r^2 = \mathrm{d}Q/\pi$ to conserve mass and ω drops to ω_1 with $\omega_1 r_1^2 = \omega r^2$ to conserve angular momentum and to regain some pressure. Then the head is both H-ΔH and
$H_1 = p_1 + 1/2\rho(u_1^2 + \omega_1^2 r_1^2)$ and $\omega_1^2 r_1^2 = (\omega r^2/r_1^2)^2 r_1^2 = (\omega r)^2 r^2/r_1^2$ (4)

Combining (2), **(3)** and (4) gives $H = p_0 + 1/2\rho V^2 = p_1 + 1/2\rho(u_1^2 + (2\Omega + \omega_1)\omega_1 r_1^2)$ (5)

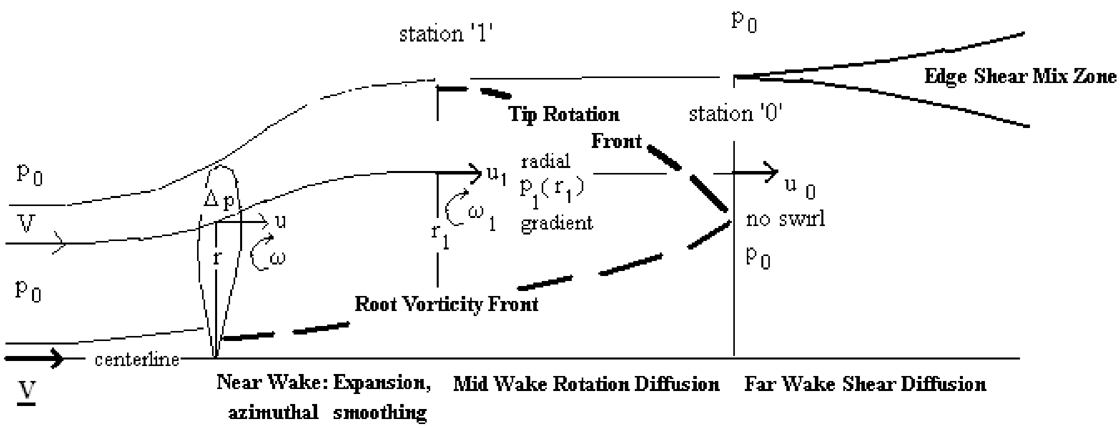
Differentiating to equate $\mathrm{d}p_1/dr_1$ to $\rho\omega_1^2 r_1$ gives $\mathrm{d}(V^2 - u_1^2)/\mathrm{d}r_1 = 2(\Omega + \omega_1)\mathrm{d}(\omega_1 r_1^2)/\mathrm{d}r_1$ (6)

Expanded Wake Swirl Models

Here Glauert's presentation of Joukowski's SET [1] is paused to add the inevitable real fluid viscous decay of this wake. Ultimately in the 'far' wake, the axial velocity deficit V-u_0 will spread into the wind, dissipating the associated kinetic energy at a further additional loss to the **wind** beyond the extracted, even Betz, power, as explicitly demonstrated by Corten [9].

The opposed axial vorticity segments will diffusively cancel faster than this spread of azimuthal ring vorticity to infinity. Also the optimum BEM axial velocity peak at the hub would tend to be decreased by the slower velocity at bigger x before both are raised by the friction with the wind outside the rotortube.

Fig 1 Idealised 3 Stage Wake



So idealise the swirl to dissipate first in isolation in a new 'midwake'. Then the suction pressure at the SET '1' would decrease to decelerate the wake flow further into a second expansion. The speed of a BEM propeller's slipstream exceeds the freestream V and has ample axial kinetic energy to regain the static pressure as the suction due to its weak swirl is dissipated. Whereas the SET optimum windmill axial velocity u_1 is about $V/3$ and the hub swirl velocity much greater and the suction much higher. Dissipation of swirl would so relieve the suction downstream that the inner axial flow would have to reverse. As in a> ½ 'over-induction', this invalidates the assumed flow picture. Thus for the windmill, SET may be only for an ideal, perfectly inviscid fluid, not a real approximately inviscid fluid.

The most optimistic but plausible model would be all the expansion with the swirl undissipated for the maximum pressure regain in the near wake, followed by swirl dissipation with no further net expansion in the mid wake. No net expansion roughly requires no average guage pressure at stage 1 to match the ambient pressure at station '0'. This 'Dissipated Expanded Swirl Head' (DESH) model should establish a more realistic upper bound on the possible rotor performance. The popular BEM is a pessimistic lower bound since its assumed induced fraction of ½ is conservative in the ring vortex picture and will be seen to imply dissipation of the full unexpanded swirl kinetic energy added by the rotor.....

So the swirl (difference) is taken to decay first, giving a state '0' with just axial velocity u_0 and ambient pressure p_0. In this first dissipation the head $½\omega_1^2 r_1^2$ of the expanded swirl is lost. Then (4) gives the simple DESH

$$dP/dQ=\Delta H=\rho\Omega\omega r^2=½\rho\,(V^2-u_0^2-\omega_1^2 r_1^2)=½\rho\,(V^2-u_0^2-\omega_1\omega r^2) \quad (7)$$

Momentum Change Equations

There is no net momentum change of the outside stream, so spanning where the rotortube outer pressure varies before the far wake, the net upwind force on the flow outside it produces in conjunction with the expansion of the edgestream must be balanced by the net downwind force from the ambient pressure p_0 over the end difference in area. (Thus the axially effective average of the edgestream pressure is p_0). Then in the rotortube momentum change between upstream at p_0, and downstream at p equal to p_0 at the rotortube edge, there is only an integral of the guage pressure p-p_0 over the downstream area. [1, p186]. Joukowski assumed the same equation applied to each annular differential area. Then

$$\text{SET}\quad \rho dQ\,(V-u_1)+(p_0-p_1)dS_1 = dT=\Delta p dS \quad (8)$$

This is not true for the DESH at '1' because its $p_1\neq p_0$ at the tip. Instead it must be applied at '0' where the pressure is constant at p_0

$$\text{DESH}\quad \rho dQ\,(V-u_0) = dT=\Delta p dS \quad (9)$$

Under DESH the head (loss) equation p_0-p_1=$1/2\rho(u_1^2-u_0)^2$ is used to link '0' and '1' and so (9) is **greater** than (8) by $\rho u_1(u_1-u_0)-½\rho(u_1^2-u_0^2)=½\rho(u_1-u_0)^2$

Combining (9) with (7) gives

$$\text{DESH}\quad VdT-dP=½\rho dQ\,((V-u_0)^2+\omega_1^2 r_1^2) \quad (10)$$

This is the power balance in the wind frame where the gross drag power of the rotor element is the useful power removed plus the wasted flux of the kinetic energy of the axial velocity deficit [6] and the expanded swirl left behind (and ultimately dissipated) in the wake.

Anyways, substituting (3) in (9)

$$\text{DESH}\quad (\Omega+\omega/2)\omega r^2=u(V-u_0) \quad (11)$$

or the orthogonality condition <u>**I**</u>•<u>***W***</u>=0 due to <u>***L***</u>•<u>***W***</u>=0 the lift <u>***L***</u> doing no work in the viewpoint of the apparent wind <u>***W***</u>. Comparing with the DESH head equation (7) expansion $r_1>r$ implies $\omega<\omega_1$ and so $u>½(T+u_0)$.

Conversely the BEM weak assumption of induced fraction of ½ is equivalent to the BEM head equation being $\Omega\omega r^2=½(V^2-u_0^2-\omega^2 r^2)$ (12) . Or the property sometimes recognised for the BEM that no swirl kinetic energy is regained in any expansion and all the original swirl head $½\rho\omega^2 r^2$ added at the rotor is ultimately lost. Thus the total BEM head loss is Δp.

However in SET, <u>**I**</u> is not proportional to <u>***L***</u> except at the tip because of p_1-p_0 terms in the momentum balance (8) at station '1'. Substituting from (5) for p_0-p_1 in (8) leads to

$$\text{SET}\quad dT=\Delta p dS=(\Omega+½\omega)\omega r^2\,dS = (\Omega+½\omega_1)\omega_1 r_1^2 dS_1+u(V-u_1)dS-½(V^2-u_1^2)dS_1 \quad (13)$$

Using $u_1 dS_1 = u dS$ gives Glauert's [1] SET equation (1.13). Solving (6) and (13) for constant circulation shows the optimum coefficient of performance to rise above the Betz limit as the tipspeed ratio is decreased [3] which is at odds with all accepted experimental data.

The DESH and SET equations yield the BEM ones for expansion without swirl, but with no expansion yet swirl, the SET, unlike the DESH, does not reduce to the BEM .....

Expansionless Suction (ST) Reduction of SET

With $r=r_1$ $\omega=\omega_1$ and $dS=dS_1$ (13) gives simply $u=½(V+u_1)$ so the variation of this fraction above ½ in SET does require expansion as well as swirl. However instead of a BEM/DESH orthogonality equation at all *x,* ST has Joukowski's (6)

$$d(a-a^2)/dx=(1+2a')d(a'x^2)/dx \qquad (14)$$

with non-dimensional local speed and interference factors, $u=V(1-a)$, $\omega=2a'\Omega$ $u_1=V(1-2a)$.

The ST simplicity gained by eliminating expansion allows comparing the BEM optimal solution with a ST optimal solution rather than the sole known constant circulation solution of the full SET. Regard *a* as a function of $a'x^2=\kappa(x)$, swirl circulation normalized by $2\pi T^2/\Omega$. By chain rule on (14)

$$(1-2a)(da/d\kappa)\, d\kappa/dx= 1+2a')d\kappa/dx(15)$$

Noting $c_p=dP/dS\,/½\rho V^3=4\kappa(1-a)$ from (1) is extreme when $1-a=\kappa da/dk$ then

$$(1-2a)(1-a)=(1+2a')\kappa(16)$$

provided $d\kappa/dx\neq 0$ and the same as for the BEM optimum [1, Equation 2.10], This implies the naive down-stream stability threshold $2xa'=1-2a$ is again reached by $x=1$. For differences δ, (14) & (16) give $(1-2a)\delta a=(1+2a')\,\delta\kappa$ and $(4a-3)\delta a=(1+4a')\delta\kappa-4\kappa^2\delta x/x^3$ so eliminating $\delta\kappa$

$\delta a=4(1+2a')a'^2x\delta x/(4-6a+10a'-16aa')$ (17) & solving (16) $4a'=[1+8*(1-2a)(1-a)/x^2]^{½}-1$ (18)

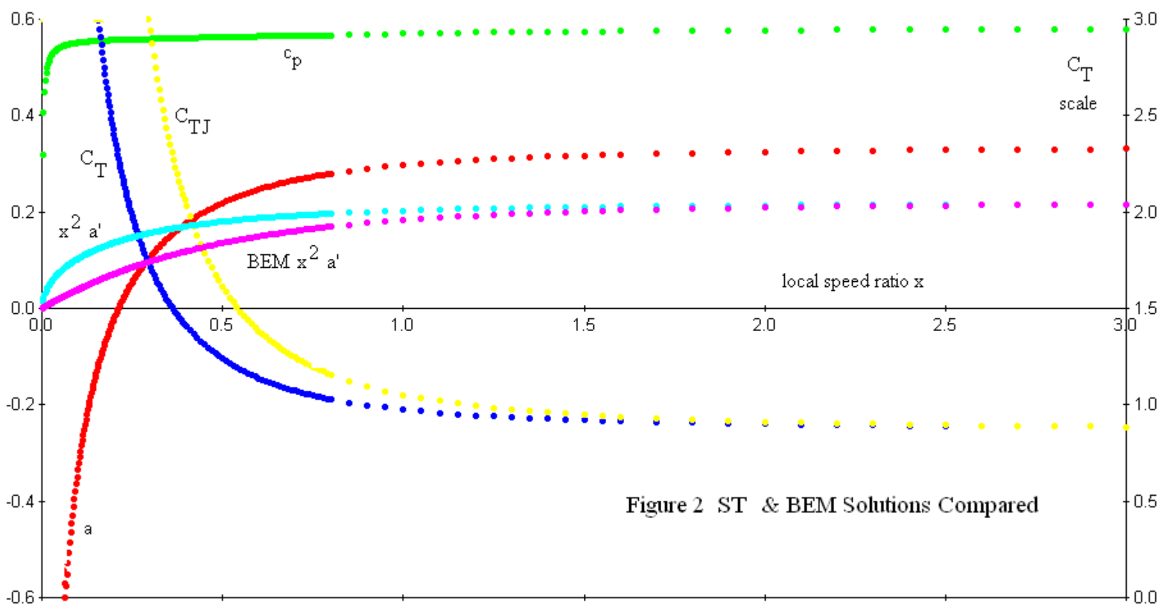


Figure 2 ST & BEM Solutions Compared

The Joukowski condition of $p_1=po$ at *X* gives from (5) the standard orthogonality $a(1-a) = \kappa(1+a')$ at the tip so the optimum there is the same as for BEM. Using (17) and (18) to march inwards from X=3, Figure 2 shows $\kappa=a'x^2$ to drop more slowly than BEM optimum x^2a' and the rotor thrust coefficient $2\Delta p/\rho V^2=c_T=4(1+a')a'x^2$ to rise singularly whilst the optimal BEM c_T fall from 8/9 to 3/4. Such high $c_T>>1$ are extremely unlikely in being above the highest rigid body C_D's≈1, an apt comparison as Medici [10] has observed rigid body type Von Karman vortex shedding by high drag rotors. The absurd *T* makes the gross windpower absorption *VT* is far greater than the undisturbed wind energy flux through the swept area! So the thrust / structural / windfarm efficiency $\eta=P/VT$ still becomes very low despite the high c_P which is virtually constant from .577 at x=3 to .557 at x=.25, before finally plummeting to 0 at

x=0. The axial interference a falls more quickly than BEM's gentle decline to ¼ at x=0 and becomes negative at x=.21 and negative singular at x=0 due to the suction being centripetally singular in the absence of any radial flow. However $p_o - p_1 > \frac{1}{2}\rho\, u_1^2$ already by x=.95 implying flow reversal downstream once the swirl has dissipated.

The other sounder full SET solution of constant circulation $\kappa = a'x^2$ and constant a satisfies (14) trivially. Again optimizing the tip c_p, these BEM tip a, κ and c_p then apply over the entire rotor, so it is the real optimum after all. Its c_{TJ}, graphed in Figure 2, rises even more quickly to be double the above at x=.25 (because the previous solution meets the ever-increasing suction as $x\downarrow 0$ by dropping a and so increasing its velocities $u=V(1-a)$ and $u_1=V(1-2a)$ instead of Δp so much). Here $u_1=v_1$ at bigger $x\uparrow 4/3$ since $u_1\uparrow \frac{1}{3}V$ & $\kappa\uparrow 2/9$ for large X, and $u=2xa'$ at x=⅔ vs .42 above. Here $v_1/u_1 \to \infty$ as $x\to 0$, so a higher critical ratio for vortex bursting will still not avoid it.

The simplicity of the $d\kappa/dx=0$ solution allows further understanding, which carries over to its solution of the general SET.[11] In its energy balance in the wind frame $dP = VdT$ less the rate of increase of the kinetic energy in the wake, the only variations from the tip values are due to ω_I, in the $(p_1-po)dS_1$ component of dT in (11) varying as $\frac{1}{2}V\rho\omega_I^2 r_1^2 dS_1$ from $dp_1/dr_1 = \rho\omega_I^2 r_1$ and in the swirl kinetic energy flux $\frac{1}{2}\rho(\omega_I r_1)^2\, VdS_1$ so these cancel perfectly (even with expansion). The logarithmically singular integrated gain due to suction in the gross power absorption by the massive drag is exactly offsetting the equally singular gain in swirl kinetic energy fed into the wake, keeping the useful $c_p(x)$ constant at the Betz limit, if the tip X is large. If the suction is reduced by <u>any</u> imperfect expansion or dissipation of the swirl, then there will be a large loss of c_p vs. this unlikely ideal cancellation.

Constant circulation means there is no vorticity except at the ends of the blades, so constant H radially. Then a dissipating stage with the DESH's swirl head loss $\frac{1}{2}\rho u_0^2 = p_1 + \frac{1}{2}\rho u_1^2 - p_0 = H_1 - \frac{1}{2}\rho v_1^2 - p_0$ gives $u_0^2 = u_1^2 + \mathcal{V}_1^2 - v_1^2$ Neglecting the weak tip $\mathcal{V}_1^2$ term makes the condition for downstream flow reversal just $u_1=v_1$, explaining the continual closeness of the swirling flow stability and flow reversal criteria. So flow reversal/instability is near x=4/3, too much to cavalierly exclude as hub, with no obvious matching model for the core.

Both ST solutions have the unacceptable drag, absurd total power, Betz limit useful power, and contradictory flow reversal features of the full SET constant circulation solution, so the conclusion is that the SET flaw is not in expansion but its swirl and suction which must burst outwards from $x\lesssim 1$.

Unlike the ST and full SET models, the DESH solution has forward flow in all wake stages, and will predict credible c_p at sensible c_T. .....

<u>Solution of the DESH Optimum</u>

With $u=V(1-A)$, $u_0=V(1-2a)>0$, $y=x^2$, & $e=y/y_1$ the DESH momentum(9) & (7) head give
$a(1-A) = ya'(1+a')$ (20) and $a(1-a) = ya'(1+ea')$ (21) so $(1-a)/(1-A)=(1+ea')/(1+a')$ (22)

And also $a(a-A)= v^2(1-e)$ (23) so $a-A$ arises only from the expansion of the swirl velocity $v= xa'$. This equation is the DESH analog of the $\mathbf{\underline{I}}\cdot(\mathbf{\underline{I}}-\mathbf{\underline{J}})=0$ in the vector generalisation of the BEM[2] . As $x\to\infty$ x^2a' must remain finite so the right hands of (20) and (21) become identical regardless of e so $a=A$ and then optimising $c_p=4ya'(1-A)$ gives $1-3a=0$ $a=A=\frac{1}{3}$ and $ya' =2/9$. This outer limit of diminishing swirl takes DESH into BEM and will dominate mass conservation to give e=½

To solve generally, follow Glauert's BEM optimisation [1] of $c_p=4ya'(1-A)$ starting with $(1-A)\mathrm{d}a'/\mathrm{d}A=a'$ (24). Equations (20) and (21) can be differentiated to give two more relating da'/dA and da/dA at a local optimum with y and e fixed, so that elimination yields a relation between the optimal A, a, and xa' which then can be solved simultaneously with (20) and (21). In detail,

$(1-A)\mathrm{d}a'/\mathrm{d}A=a'$ (24) $(1-A)\mathrm{d}a/\mathrm{d}A-a = y(1+2a')\mathrm{d}a'/\mathrm{d}A$ (25) $(1-2a)\mathrm{d}a/\mathrm{d}A=\mathrm{y}(1+2ea')\mathrm{d}a'/\mathrm{d}A$ (26)

Subtracting $(1-2a)$x(25) from $(1-A)$x(26) eliminates da/dA ,

$a(1-2a)=\{2a'(2a-eA)+(2a-A)+2a'(e-1)\}\, y\mathrm{d}a'/\mathrm{d}A$ (27)

Multiplying (24) by (27) by $ya'(1+a') =a(1-A)$ (29) allows cancellation of $ya(1-A)a'\mathrm{d}a'/\mathrm{d}A$ so

$(1-2a)(1+a')=\{\ \}$ (29) so solving $a'(6a+2e(1-A)-3) =(1-4a+A)$ (30)

which reduces to the standard $a'=(1-3a)/(4a-1)$ for the BEM $A=a$ and $e=1$.

Solving for a' in (22) $a-A=a'(1-e(1-A)-a)$ (31) so multiplying (30) by (31) eliminates a' to give

a quadratic in a and A. $(a-A)(6a+2e(1-A)-3) =(1-4a+A)(1-e(1-A)-a)$ (32)

$2a^2+ a(2-2e(1-A)-5A) +e(1-A)^2+2A-1 =0$ (33) With $t=1-A$ $et^2+t(a(5-2e)-2)+(1-a)(1-2a)=0$ (34)

The equations suggest a Glauert-type 'back substitution' solution such as supposing an a and e, solving for t ie A(- root), then a', and finally for x. Equations (30) and (31) are both needed for a', because (31) is indeterminate (0/0) in the e=1 A=a BEM limit, whereas at e=¾ equation (34) exactly solves as $a=(1+A)/4$ which makes (30) indeterminate.

The determination of e involves more complication than just awkward coupling integration out from x=0, because e is the local expansion ratio to station '1' not '0'. (Using it as the latter is equivalent to collapsing the two stations together in a suctionless $p_1=p_0$ reduction of SET). Even though there is no rotortube expansion between these two of the same average pressure, there is local variation in the expansion because the centreline fluid decelerates from suction at 1 to ambient pressure at 0, whilst the outer half of the fluid has balancing positive pressure at 0 and so accelerates.

The strategy adopted here is to try a variation of e with a (to ultimately e=½ at a=A= ⅓) from one of these starting values and then integrate for the actual expansion ε from the solution, and try to adjust the e's to match ε. To specify the equations used to calculate ε, at the end of the expansion, $v_1/V=2xa'\sqrt{e}$ $dp/dx_1=\rho v_1^2/x_1$ so integrating by parts for no net pressure gives for $\mathrm{p}_1=(p_1-p_0)/\tfrac{1}{2}\rho V^2$, $y_1=y/e$

$\mathrm{P}_1\, Y_1 =\int(v_1/V)^2\, dy_1$ and then $\mathrm{p}_1=P_1- \int(v_1/V)^2\, dy_1/y_1$ $u_1/V=\sqrt{(1-2a)^2-\mathrm{p}_1}$ $y_1/\varepsilon= \int(1-A)V/u_1\, dy$ (36)

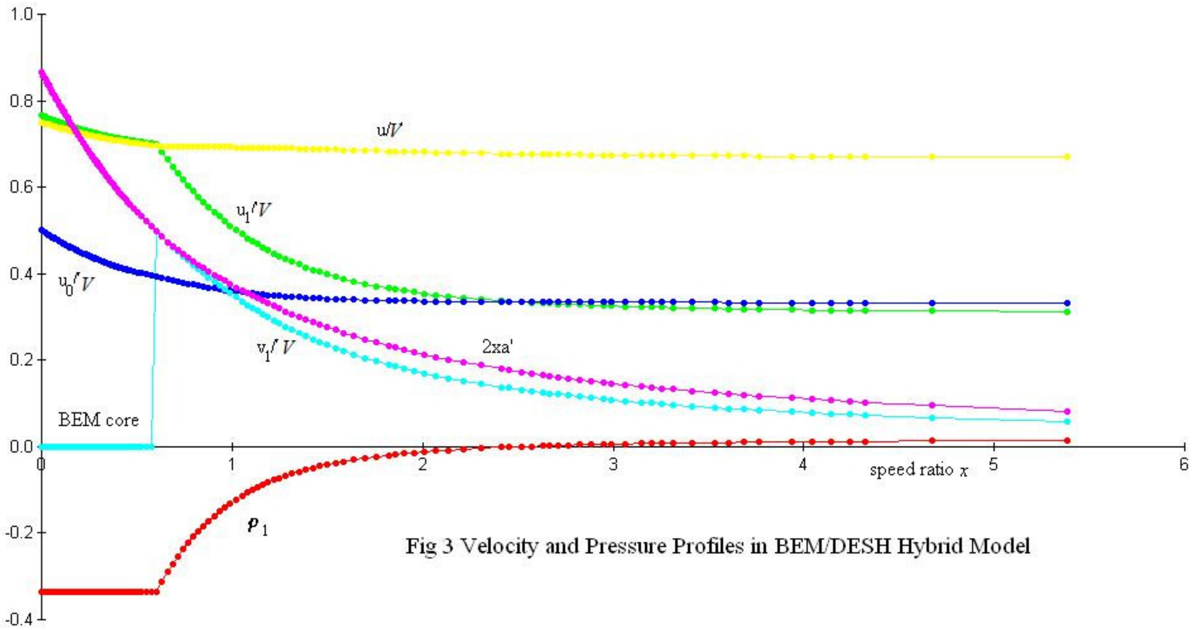


Fig 3 Velocity and Pressure Profiles in BEM/DESH Hybrid Model

where P_1 and Y_1 are the tip values. Given that A generally increases with x, u_0 is smaller at the tip so for the same pressure deviation from p_0, $|u_1-u_0|$ will be greater on the outer half of the rotor, so there will be a slight net recontraction from 0 to 1. This could be iterated out by dropping P_1 slightly. Generally the difference between e and ε can be eliminated except in a region of small $x<1$ where ε invariably exceeds 1 due to the strong u_1 at small x_1 due to the (singular) suction there.

Hybrid Optimal BEM-DESH

This suggests a hybrid model (Figure 1) which ignores the swirl and radial pressure gradient at '1' in a BEM core zone on the conjecture that the clearly unstable swirl near the rotor axis is already dissipated or exported by vortex bursting. This very much limits the small x rise of u_1 and ε (into contraction) so that e and ε are essentially matched at 1 in this BEM core zone, and the transition to non-zero v_1 and DESH then can be made at x=.57. The expansion of the core region is then delayed to between stages 1 and 0, when the interface suction can decline with the dissipation of the outer swirl. This agrees with CFD calculations showing about a 1 diameter long zone downwind of a streamlined nacelle before the central wind diminishes[12]. The higher 'turbulence' intensity level in the CFD wake suggests an instability such as vortex bursting.

The small angular momentum of the BEM core at the rotor must transfer outwards in the burst, but its small addition to the DESH zone swirl will be ignored, just as has any radial transfer of axial momentum always been. It should be emphasised that the features of interest develop as e drops significantly into expansion typically at x>1. Figure 3 shows the velocities at all three stations as well as p_1, the non-dimensional pressure at station 1. (The ST solutions pass through the centerline peak here of p_1 of -.336 at x=.67 or .72 for $d\kappa/dx=0$ and neither ST nor SET can be matched to a BEM core) Note how slight the p_1 is on the tip streamline. Here u_1 exceeds v_1 everywhere by a minimum of 40% stability margin downwind of the rotor. This happens to coincide with the .6 upper limit of tangential to axial velocity observed by

Masouh[8]. Medici [10] measured a broad midblade peak of swirl velocity at 1 m/s with axial velocity 4m/s under-reduced from the true wind 8m/s at the same 1 diameter downstream with only weak rotation near the axis. There is simply no evidence of the heavy middle wake swirl predicted by SET.

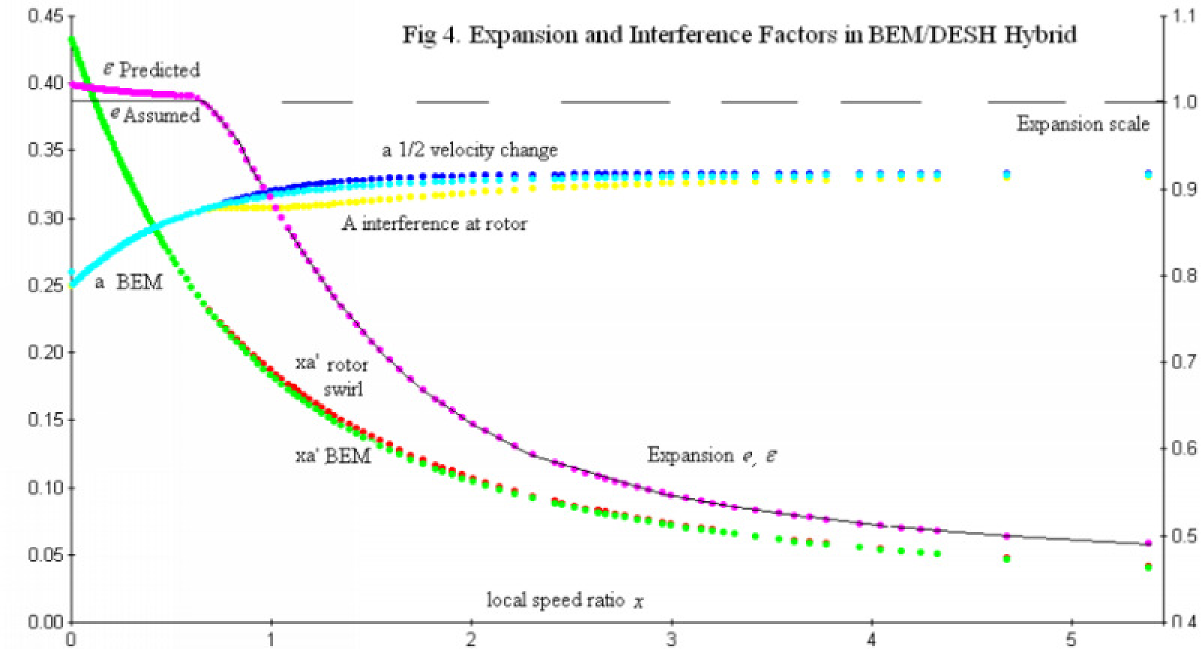


Fig 4. Expansion and Interference Factors in BEM/DESH Hybrid

Fig 4 shows the main differences in the rotor flow vs the all-BEM optimal are the increase in *xa'* by about 3% due to the lower swirl penalty and lowering of the interference *A* at the rotor by about .013 peak whereas *a*, half the wake net velocity change only rises slightly. So the expansion from far upwind to stage 0 is virtually unchanged , but slightly more of this occurs downstream. The difference *a-A* is about .02 reducing to zero at the tip consistently with the vortex picture, and close to CFD[13] and ring vortex [4] calculations for constant A=⅓. The changes raise the thrust coefficient $c_T=4a(1-A)$ over the BEM by a peak of 3.2% at about x=1.35, to an absolute peak of 905 at x=1.6 vs the BEM asymptote of .89.

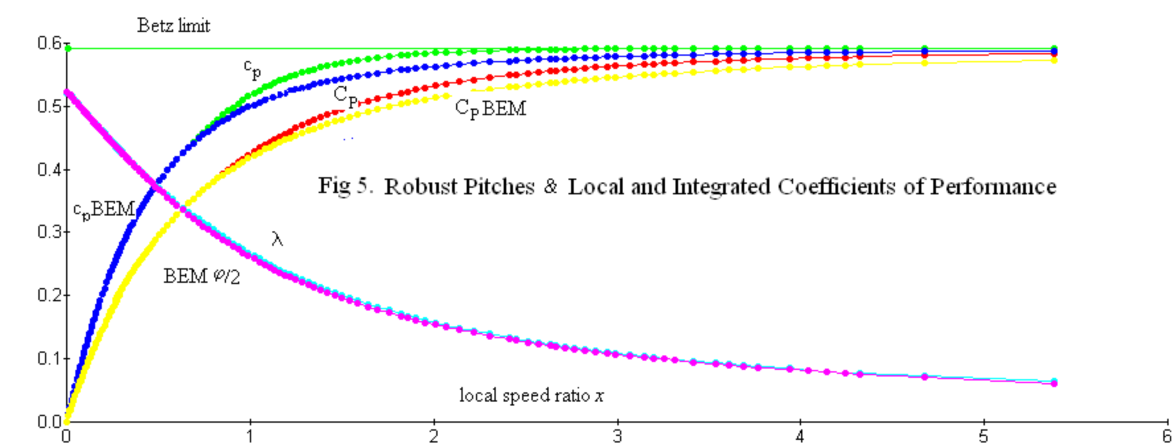


Fig 5. Robust Pitches & Local and Integrated Coefficients of Performance

Figure 5 shows the difference in local c_p from the Betz limit is more than halved from x=1.5 on and the greatest absolute difference of c_p is about .025 at x=1.5. Of this 4.6% increase, 3% comes from the swirl increase directly and 1.6% from the increase in *(1-A)*, the flux at the rotor. The net improvement in local thrust efficiency is about 1.5% peak. The local power virtually obtains the Betz limit by x=2. These results are for X=5.5, but values at a given x are only weakly affected by X, e.g. $c_p(1.5)$ is increased by only .3% for X=3.5. So it is a good approximation to just integrate the $c_p(x)$ at middle X=5.4 to get $Cp(X)$. An increase peaking at about .019 or 3.5% is seen to broadly span X=1.5 to 3. There is no question of the $c_p(x)$ curve being shifted so much above the BEM $c_{pB}(x)$ curve that the latter is closer let alone identical to the new $Cp(X)$ curve as in the ST model.

Robustly Optimal DESH Blade Elements

The lift on the B blades each of chord c for solidity $\sigma=Bc/2\pi r$ must provide the velocity change $2\underline{\mathbf{I}}$ with components $2aV$, and $2v=2a'xV$ across the rotor, again assuming there is not net pressure force on each annular streamtube as there isn't on the entire rotor streamtube. If $\underline{r}$ is the unit radius, $\underline{\mathbf{W}}$ is the apparent wind, magnitude W, angle φ to tangential

$$\tfrac{1}{2}\rho\, W C_L\, \underline{r} \times \underline{\mathbf{W}}\, \sigma ds = -d\underline{\mathbf{L}} = 2\rho \underline{\mathbf{I}}\, (\underline{\mathbf{W}} \bullet \underline{\mathbf{V}})\, ds/V \quad \text{the “ blade element” equation} \quad (37)$$

Thus with $\varpi=\pi\sigma/2 = Bc/4r$, $\varpi C_L/2\pi = \tfrac{1}{4}\sigma C_L = I\sin\varphi/W = v/W$ (38)
Call v/W, f. For optimal BEM $f=1-\cos\varphi$; so with $k=\Omega c/T$, then $BkC_L/2\pi=4xf$ (39)

W is composed from the induced flow components A and xa' as $(W/V)^2 = (1-A)^2 + x^2(1+a')^2$. The increase in swirl increases v/W so in Figure 6 the optimum $BkC_L/C_{LS} = 8\pi xf/C_{LS}$ is increased by about 2.5% with the peak difference at x=1.5, where subscript s denotes stall value.

Prior to stall (38) is $\varpi\sin\alpha = f$ where the (¾ chord) angle of attack is α *is* φ the angle of W minus λ, the blade pitch angle. Versus the BEM optimum, the DESH increase in 1-A dominates the increase in 1+a' so $\tan\varphi = (1-A)/x(1+a')$ is increased by peak 1%.

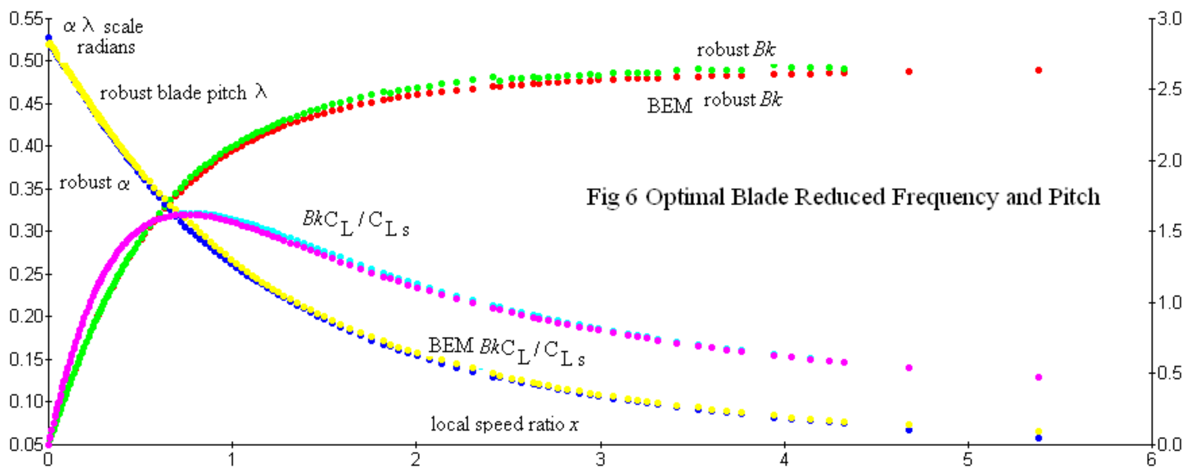


Fig 6 Optimal Blade Reduced Frequency and Pitch

Then, for a blade element to robustly remain optimal at its fixed $\Omega, c, r, \varpi, \lambda$ as the windspeed varies the angle φ of W, requires $\varpi\cos(\varphi-\lambda) = df/d\varphi$. Dividing gives $\tan(\varphi-\lambda) = f\, d\varphi/df$ (40), differentiated numerically on the spreadsheet of optimal f and φ vs x. Then (38) can be solved for *Bk* as graphed versus the simple BEM robust $\lambda=\frac{1}{2}\varphi$, $\varpi=2\sin\lambda$ [2]. Figure's 5 and 6 show that relative to BEM, the DESH increase in BkC_L/C_{LS} would be met by a proportionate rise in chord *Bk* because the angle of attack $\alpha=\varphi-\lambda$ is little changed as the slight growth in φ is matched by the change in robust pitch λ. So the design robust chord and pitch are bigger. With $\alpha_S=20^o$=.35 (enhanced over static by the centrifuging in the boundary layer) in Figure 6, the plots of the robust *Bk* falls slightly below the plots of BkC_L/C_{LS} implying slight jogs in the best blade profiles on account of stall. (A super-enhanced stall of 26^o would make the robust rule for all *x*). In reality bending strength will require bigger hub thickness and so chord for cantilever blades, as well as curtail the outer chord and pitch to limit the absolute torque at high *T* and low *X*.

The drag correction to the BEM [14] does reduce the robust chord *Bk* slightly with large *x*, before the Prandtl correction[1,14] rounds off the tip and also does lowers its robust pitch λ.

Conclusions

Generally the BEM optimal blade elements are not changed much with the DESH best case expansion. Rather than try to detect the modest increase in mid-*x* power in always difficult measurements of the larger net rotor C_P a simpler way to find the more accurate design model might be to measure centreline pressure and velocity behind a Hawt rotor, ideally in a low blockage long windtunnel. In the pure BEM there could be a rapid recovery from -5/16 dimensionless suction immediately behind the rotor back and doubling of the axial velocity defect, if the entire swirl is dissipated in one all-encompassing vortex-burst to large scale turbulence. In the above hybrid model, after an initial smaller burst, the central axial velocity is close to the BEM **rotor** value for considerable distance downstream until the rotor streamtube has expanded, and only then does the core gradually expand and slow down at the expense of the tip streamtubes, whilst the non-dimensional guage pressure slowly rises from -.34, as the outer swirl slowly dissipates. (With SET one would see a much heavier pressure drop at the rotor hub with a permanent residual downstream.)

Existing experiments have been guided by BEM patching and preoccupations like tip vortices, so remarkably none seem to have made these pressure measurements, let alone examine the swirl at multiple downstream stations to track its expansion and dissipation.

In summary, the BEM model is best justified by supposing the loss of all the swirl energy added at the rotor by vortex bursting and turbulent diffusion. It is judged to rule at the root and at a high speed ratio tip. However for intermediate *x*, swirl expansion with some pressure regain before dissipation may be possible by crude stability measures, providing reduced induced axial backflow and a positive power correction to the standard BEM model at mid blade.

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