Vector Induced Velocity BEM VAWT Theory

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Abstract: *Double Multiple Streamtube 'DMST' models of the Vertical Axis Wind Turbine 'Vawt' take its interference as unidirectional flow against the wind. Whereas in the standard blade element momentum 'BEM' theory of Horizontal 'Hawt's', the interference velocity is proportional to the lift vector normal to the apparent wind. High design tip speed ratio brings the apparent wind to tangential, making the BEM interference flow radial across the Vawt blades. Not even upwind on average are the lateral drift induced by the Vawt bound vorticity and the reaction velocity to the torque in Hawt theory. A new vector BEM formulation includes such crosswind Vawt interference. For the two Vawt passes through the wind it predicts a lower and narrower operating peak of tangent and 'passive pitch' Vawts than the DMST or Hawts. But it gives the pitch cam shape for broadening the operating peak and benignly avoiding high downwind drag and reverse wake flow. The Prandtl tip correction is included and compared to isolated wing aspect ratio factors.*

Keywords: Vertical Axis Wind Turbine, Blade Element Momentum, Darrieus, cyclic pitch, passive pitch, Double Multiple Streamtube, robust design

1. INTRODUCTION

Darrieus patented the Vertical Axis wind Turbine (Vawt) in the 1920's whilst in 1935 Glauert [1] derived the theory of horizontal axis wind turbines (Hawts) still used today. The Canadian government reinvented the 'eggbeater' tangent blade Vawt in the 1960's and has only ever funded its R&D, not even any other Vawt. Sandia labs, but not the entire DOE, also specialised on the troposkein Vawt in the same period, leaving better documentation on the Internet before jumping to Hawt R&D today.

The Vawt analysis **in isolation** by such proponents began with windwise induced flow of Froude's 1D actuator theory [1] which is actually the high speed ratio limit [2] of Glauert's Hawt, now solved analytically in general. This new exact solution[2] shows that the true limit in the Vawt case of blades moving obliquely to the wind is the induced flow being locally normal to the path and span, and only upwind when averaged over the sweep.

Wilson & Lissamen [3] averaged between the upstream and downstream Vawt sweeps through each streamtube as if the lateral flows induced by the upwind and downwind cuts were equal and opposite and cancelled each other out at both cuts. This would be valid for a sufficiently elliptic blade path. Then the BEM finds that the common upwind induced flow varies as the sine of the azimuthal angle.

Lasaukas [4] & Paraschiviou [5] split this sinusoidal axial induced flow 3:1 'not commonly' in favour of the lee pass, yet still neglected any crosswind induced flow anywhere on either semicircle. This "double multiple streamtube" (DMST) Vawt became the foundation on which big computer code superstructures grew, their very complexity obscuring its soundness and its relation to Hawt theory, undermining rational comparison of the two configurations.

The aim of this paper is to formulate a Hawt-consistent BEM vector double pass model, solve it analytically for 2 and 3D Vawts, and so compare Vawts with Hawts with as common, transparent, and unbiased a theory as possible.

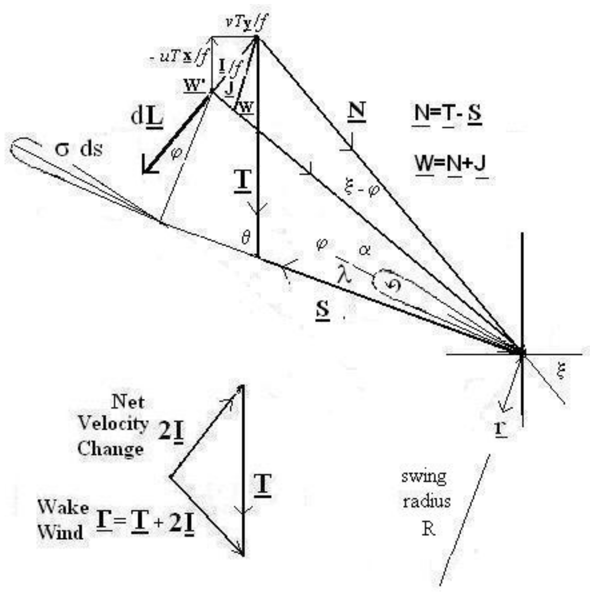

Figure 1 Windward Pass 2D Vawt

2. THE GENERAL VECTOR BEM EQUATION AND POWER RELATION

As in Fig 1 consider a section of a continuum of minute chord blades of solidity σ in a true wind **T**.

The blades move at velocity $\underline{\mathbf{S}}=XT\,\underline{\mathbf{s}}$ at speed ratio X times the windspeed 'T', along a surface with downwind normal $\underline{\mathbf{r}}$ at standard Vawt angle $\pi/2-\theta$ to the true wind **T**, so the Naive, Nominal or No-lift apparent wind is $\underline{\mathbf{N}}=\underline{\mathbf{T}}-\underline{\mathbf{S}}$ at angle 'ξ' to the path. Let $2\underline{\mathbf{I}}$ be the velocity change the blades produce downstream, and their ¾ chord angle of attack be 'α' to the real apparent wind $\underline{\mathbf{W}}$ at non-zero $\underline{\mathbf{I}}$ at angle 'φ' to the path of radius R.

Hawt BEM theory ignores any net pressure force on the stream tube walls and ends. (Detailed analysis [6] shows this is equivalent to assuming that all the crosswind velocity added at a Hawt rotor is dissipated in the wake.) It opposes the rate of change of momentum of the fluid density ρ passing through spanwise by movement area ds, to the Joukowski airfoil lift 'd$\underline{\mathbf{L}}$', from the net blade area, σds

$$\rho\ ds\ (\tfrac{1}{2} l\sigma W C_L)\ \underline{\mathbf{z}} \times \underline{\mathbf{W}} = -d\underline{\mathbf{L}} = 2\,\underline{\mathbf{I}}\,\rho\, ds\ \ \underline{\mathbf{W}}\bullet\underline{\mathbf{r}} \quad \textbf{the BEM equation} \qquad (1)$$

The bracket is the bound vorticity linear density Γ along ds , so the spanwise unit vector $\underline{\mathbf{z}}$ takes its sign by the right hand rule out of the page here on the windward pass. Now

$$\underline{\mathbf{W}}\bullet\underline{\mathbf{r}} = W\sin\varphi \ \text{ so } \ \underline{\mathbf{I}} = E\,(\underline{\mathbf{z}} \times \underline{\mathbf{W}}) \ \text{ so } \ \mathrm{I} = E\,\mathrm{W} \quad \textbf{the tangent blade BEM equation} \qquad (2)$$

where E is the circulation divided by twice the normal velocity or $\Gamma/\ 2\underline{\mathbf{W}}\bullet\underline{\mathbf{r}} = \sigma C_l/\,4\sin\varphi$. For small α $C_L = 2\pi\ l \sin\alpha$, with lift slope factor l and then E is the (tangent) Effective solidity $\pi\sigma le/2 = el\varpi$ where $e=\sin\alpha/\sin\varphi$, and ϖ is the solidity factor $\pi\sigma/2=Bc/4R$ for B blades each of chord c. If the ¾ chords are tangential to the path as in the **fixed tangent blade** Vawt $\alpha=\varphi$ so e=1 & $E=l\varpi$.

To aid self-starting, "**passive pitch blades** "[4,7-9] use a balance between the centrifugal blade pitch moment varying as $(XT)^2\lambda$ due to a counterweight on an outboard arm to the blade, and the lift moment varying as $W^2\alpha$. Since the blade pitch from tangential $\lambda=\varphi-\alpha$ and $W\downarrow XT$, ignoring blade inertia and unsteady aerodynamics, their e is a constant fraction at all θ and large X. The Vawt analysis here for constant E with θ will apply thirdly to the optimum design at X_d of the **robust cyclic pitch** Vawt[2] where e_d=½.

The first two e's and so E's are further invariant with X to signal narrow power bands, because the power optimum I is about $T/3$, but for tangent blades I changes almost linearly with its rotational speed Ω. That is when $W \to XT$, the interference I/T tends to the induction $'a_0' = X\,E = Xel\varpi = Xel\,Bc/4R$. With angular velocity Ω, $X=\Omega R/T$, so π and the distance to the axis 'R' of the rotor segment cancel and $a_0 = ½BKe\,l$ where $K=\Omega c/2T$. Chaney et al[10] saw that this cancellation allows analysis of a 'spherical' Vawt of constant chord blades of constant radius r (ignoring that the speed ratio must drop to zero towards the poles.) Forming the dot product of **I** with the unit tangent **s**, the triple product identity gives

$$- \mathbf{\underline{I}} \bullet \mathbf{\underline{s}} = - E\,\mathbf{\underline{W}} \bullet (\mathbf{\underline{s}} \times \mathbf{z}) = E\,\mathbf{\underline{W}} \bullet \mathbf{\underline{r}} = \Gamma/2 = E\,W \sin\varphi \qquad (3)$$

Thus the component of velocity change opposite to the motion is half the circulation and E times the normal velocity across the path. From the RHS of (1), the component of lift on the blade's area $\sigma\,\mathrm{d}s$ in the direction of travel is proportional to $- \mathbf{\underline{I}} \bullet \mathbf{\underline{S}} = a_0\,\mathbf{\underline{W}} \bullet \mathbf{\underline{r}}$ and so the useful power contribution dP from ds is

$$\mathrm{d}P/\mathrm{d}s = \mathrm{d}\mathbf{\underline{L}} \bullet \mathbf{\underline{S}} = \rho XT\,\Gamma\,\mathbf{\underline{W}} \bullet \mathbf{\underline{r}} = 2\rho T a_0\,(\mathbf{\underline{W}} \bullet \mathbf{\underline{r}})^2 \qquad (4)$$

At high X, the normal component of the lift force is similiarly $2\rho T\,a_0\,\mathbf{\underline{W}} \bullet \mathbf{\underline{r}}\,\mathrm{d}s$ [10]. Here the last squaring reflects the Vawt generating positive lift power by $\sin\varphi$ thrust resolution of its sin α lift. If **D** is the net flow relative to the ground, $\mathbf{\underline{D}}=\mathbf{\underline{W}}-\mathbf{\underline{S}}$ and since $\mathbf{\underline{I}} \bullet \mathbf{\underline{W}}=0$ the first part of (4) gives $\mathrm{d}P/\mathrm{d}s = 2\rho(\mathbf{\underline{W}} \bullet \mathbf{\underline{r}})\mathbf{\underline{I}} \bullet \mathbf{\underline{D}}$ Equate this to the loss in flow kinetic energy as it crosses the rotor segment, again ignoring any pressure deficit in the wake. The change in velocity squared is $\mathbf{\underline{T}} \bullet \mathbf{\underline{T}} - (\mathbf{\underline{T}}+2\mathbf{\underline{I}}) \bullet (\mathbf{\underline{T}}+2\mathbf{\underline{I}}) = -4\mathbf{\underline{I}} \bullet (\mathbf{\underline{T}}+\mathbf{\underline{I}})$, so $\mathrm{d}P/\mathrm{d}s = 2\rho(\mathbf{\underline{W}} \bullet \mathbf{\underline{r}})\mathbf{\underline{I}} \bullet (\mathbf{\underline{T}}+\mathbf{\underline{I}})$ also so $\mathbf{\underline{I}} \bullet (\mathbf{\underline{D}}-\mathbf{\underline{T}}-\mathbf{\underline{I}}) = 0$. Thus the component of the average induced flow $\mathbf{\underline{J}}=\mathbf{\underline{D}}-\mathbf{\underline{T}}$ in the direction of the velocity change 2**I** must be **I**, as in Fig 1. So the difference **J**-**I** in the cross-sectional plane is parallel to **W** . Call the apparent wind based on **I**, , **W'** which is thus in the same direction φ as **W**, but of different magnitude by $O(a_0T)$. Since the norm of **J**-**I** should be less than ½ I or $T/6$ and $W \approx XT$ for $X>2$ it will be a good $O(ma_0)$ approximation to replace **W** by **W**' in (2)

Prandtl [1] considered the induced flow **field** around the vortex sheets trailing at W from the blades and thus spaced at $t = 2\pi \sin\varphi\, r/B$. They move downstream at self-induced 2**J** perpendicular to **W**. He found the induced flow **J** at the blades to be **I**/f , $f \lesssim 1$ a function of the distance from the blade tip relative to s. This reduces the effective length of a straight Vawt blade by .442t.

3. GENERAL SOLUTION FOR BEM HAWT AND VAWT WINDWARD PASS

Substituting for $\underline{\mathbf{W}}$ in (2) as $\underline{\mathbf{N}}+ \underline{\mathbf{I}}/f$ and then for this $\underline{\mathbf{I}}$ by (2) again

$$\underline{I}=E\,\underline{\mathbf{z}} \times (\underline{\mathbf{N}} + \mathbf{I}/f) =E\,\underline{\mathbf{z}} \times \{ \underline{\mathbf{N}} +E\,\underline{\mathbf{z}} \times (\underline{\mathbf{N}} + \mathbf{I}/f) /f\} =E\,\underline{\mathbf{z}}\times\underline{\mathbf{N}} - E^2(\underline{\mathbf{N}} + \mathbf{I}/f) /f \quad (5)$$

So with $H=N^2/W'^2 =1+ E^2/f^2$ the composition of $\underline{\mathbf{I}}$ perpendicular and parallel to the known $\underline{\mathbf{N}}$ is

$$H\mathbf{I}=E\underline{\mathbf{z}} \times \underline{\mathbf{N}} -E^2 \underline{\mathbf{N}} /f =-a_0T\,\underline{\mathbf{r}}+E\underline{\mathbf{z}}\times\underline{\mathbf{T}}+Ea_0T\underline{\mathbf{s}} /f -E^2\underline{\mathbf{T}}/f \quad (6)$$

With the true wind $\underline{\mathbf{T}}$ in the $\underline{\mathbf{x}}$ direction at standard 2D Vawt angle $\pi/2-\theta$ to $\underline{\mathbf{r}}$, dividing by T gives the non-dimensional Cartesian components u,v of $\underline{\mathbf{I}}/T$ as

$$Hu= a_0\sin\theta +E^2/f +Ea_0\cos\theta /f \quad Hv= a_0\cos\theta +E -Ea_0\sin\theta/f \quad (7)$$

It is at first surprising that at fixed a_0, as speed ratio $X\rightarrow \infty$ and $E\rightarrow 0$, v does not vanish; but rather the net induced velocity becomes - $a_0\,T\,\underline{\mathbf{r}}/f$ directed outwards against $\underline{\mathbf{r}}$ (and so does not involve any torque for lack of angle). This is a direct consequence of it and the lift having to be perpendicular to the blade apparent wind which high X brings towards tangential. Whilst the angle of attack and d$\underline{\mathbf{L}}$ decrease towards the sides, so does the normal flux, so from (1) the induced component stays constant.

The term $v=E(1 - a_0\sin\theta/f)$ is productive crosswind induction from the true wind component of $\underline{\mathbf{W'}}$ reacting to the torque , in the Hawt $\theta=\pi/2$ $v\rightarrow 2/9X$ at a_0=⅓ like Xa'. In the 2D Vawt $v=E$ can also be considered as the BEM approximation of the crossflow induced by the bound vorticity strength $2EW\sin\varphi \approx 2ET\sin\theta$.

In the first half of (6) the $E^2\,\underline{\mathbf{N}}/f$ terms will reduce the apparent wind $\underline{\mathbf{W'}}$ component in the $\underline{\mathbf{N}}$ direction to $\underline{\mathbf{N}}/H$ whilst the orthogonal term $E\underline{\mathbf{z}}\times\underline{\mathbf{N}}\,/fH$ rotates $\underline{\mathbf{N}}$, both reducing $\underline{\mathbf{W'}}\bullet\underline{\mathbf{r}}$ below $\underline{\mathbf{N}}\bullet\underline{\mathbf{r}} = \underline{\mathbf{T}}\bullet\underline{\mathbf{r}}$. Thus

$W'\sin\varphi = \underline{\mathbf{W}}'\bullet\underline{r} = (\underline{\mathbf{T}}\bullet\underline{\mathbf{r}} - T\,q_0 - E\underline{\mathbf{T}}\bullet\underline{s}/f)/H$ (8)

with $a_0/f = q_0$ just the total induction or induced flow/mW at the blade. W' is most easily formed from

$$W'^2 = N^2/H = T^2(X^2+1+2X\cos\theta)/H \quad (9)$$

so $XT\sin\varphi$ approximates as (8) plus $m\cos\theta$ and m^2 terms, where $m=1/X$ so $E=ma_0$.
If ξ is the no-lift apparent wind $\underline{\mathbf{N}}$ angle Fig 1 and $m\,q_0=a_r\xi$,

As $m\to 0$ $\xi\to m\sin\theta$ $\varphi\to\xi - m\,q_0 = \xi(1-a_r)$ $dP/ds \to 2f\rho(\underline{\mathbf{T}}\cdot\underline{\mathbf{r}})^3\{(1-a_r)^2 a_r\}$ (10)

optimum at the Betz limit 16/27 of $\frac{1}{2}f\rho T^3\sin^3\theta$ when $a_r=\frac{1}{3}$ For an optimal 2D Vawt (implicit variable pitch), the integration over θ gives a best single pass $C_{P2}=P/\frac{1}{2}\rho T^3(2R)$ of $2f/3$ the Betz limit or $.395f$ [2]. Note then f is averaged over the variable t of the path . From (8) , the exact power (4) for any m is

$$dP/ds = 2\rho f q_0 T^3(\sin\theta - q_0 - mq_0\cos\theta)^2/H^2 \quad (11)$$

which led to a cubic equation in q_0 for the optimum with a triple angle solution [2,12] and the discovery that in general the best dP/ds comes from $\varphi=2\xi/3$ by choosing $E/f=mq_0=l/fW=\tan(\xi/3)$.

For constant e tangent $\lambda=0$ or passive pitch Vawt blades, mq_0 must instead be constant whilst θ varies ξ, with φ just the difference, so q_0 must optimise the integrated power. By averaging (11) over $\theta=0°\text{-}180°$ with 2D

$$ds_2=Rd\theta \quad C_{P2}=\pi f q_0\{1-8q_0/\pi+2q_0^2+g^2/f^2)/H^2 \quad (12)$$

with the approximate maximum at small m of $C_P=.388f$ at $q_0=.265$. This non optimally "over-induces" on the sides making the net flow tangent to the path to end the pass or $\varphi=0$ at $q_0=\sin\theta_e$. Taking $\sin\theta\approx\theta$ gives a correction $-4q_0^4/3$, weak at $q_0=.265$ for $C_P=.381f$ but rising very fast with q_0.

Neglecting the E term in (8) as $m \downarrow 0$, for the Vawt sphere radius r, the normal radial velocity is axisymmetric about the wind direction at generalised angle θ

so $ds_3 = r\cos\theta\,\mathrm{d}\theta$ & $C_{P3} = P/\tfrac{1}{2}\rho T^3(\pi r^2) = 8a_0\,(1/3 - a_0 + a_0^2)$ (13)

which peaks at 8/27 or half the Betz limit at $a_0 = 1/3$.

These BEM feathered lee pass Vawt C_P can be expected to be conservative because considering an off-center streamtube, the more downwind intersection and consequent pressure drop of the outer diverging streamline with the half actuator than the inner gives a net downwind pressure force on the streamtube neglected in the BEM. With a lee drop the effect is reversed so for the complete Vawt's here, the BEM should be a good approximation.

As the windspeed varies slightly away from design for general $\xi(\theta)$, e.g. including a leeward pass, to robustly maintain the same design $a_r = 1 - \varphi/\xi$, and so design C_P with a fixed pitch cycle $\lambda(\theta)$ requires that the high X limit of (1) $\varpi(\varphi - \lambda(\theta)) = (\xi - \varphi)\,\varphi$ be invariant to small changes in $\xi(\theta)$ with m about $\xi_d(\theta)$, so differentiating it with respect to φ as proportional to ξ gives $\alpha_d\,(\theta) = \lambda(\theta) = \tfrac{1}{2}\varphi_d$ so $e_d = \tfrac{1}{2}$ for all θ and

$\varpi = 2(\xi_d - \varphi_d) = 2m_d q_{0d}$. These are respectively half and twice the tangent blade values $\alpha = \varphi$, $e = 1$, and $\varpi = \xi_d - \varphi_d = m_d q_{0d}$, so analysis of the tangent blade will provide the pitch cycle $\lambda(\theta)$ for broadening its operating peak from quadratic to quartic in m with a_0 changed from linear in X to quadratic benignly peaking at a_{0d} at design m_d [2].

4. WILSON'S VAWT COMMON INDUCED FLOW AND VORTEX MODELS

For the complete Vawt, Wilson & Lissamen [3] very simply ignored the separation of the windward and leeward passes so they had the same BEM induced flows at each other, with cancelling signs of v, so the net $\underline{\mathbf{D}}$ is then just $(1-2u)T\,\underline{\mathbf{x}}$, changing 2D solutions (7) (for $0<\theta<\pi$) to $u = 2a_0\sin\theta$ and $v = \pm a_0$

$cos\theta \pm (1\text{-}2u)E$. This optimistic assumption also ignored the vertical axial induced flow with the troposkein shape as averaging to zero, and resolved the blade span into vertical segments.

The power is still $2\rho(\underline{\mathbf{W}}\bullet\underline{\mathbf{r}})\underline{\mathbf{I}}\bullet(\underline{\mathbf{T}}+\underline{\mathbf{I}})$ so defining the common induction factor $A=2q_0$ and double underlining to mean average value over θ

$$C_{P2}=4\pi f q_0 \; \underline{\underline{(1\text{-}2q_0 \sin\theta)^2 \sin^2\theta}} \; = \; \pi f A(\; 1 \; \text{-}16A/\, 3\pi + \, 3A^2 /4 \;) \tag{14}$$

which peaks broadly at $.554f$ at A=.401 with the central interference perilously close to the value of ½ at which the centreline velocity in the wake is reversed and the model collapses. With Giromill type variable pitch for ideal constant α and Γ the induced axial flow is constant aT with θ, and C_P the Betz 2D $4(1- a)^2\, a$. Indeed with Wilson's simple 2D vortex wake picture [5,11] for constant Γ, (4) shows that the power for **any** pass is **exactly** $XT\,|\Gamma|$ times the **net** flux which is equatorially $2rT(1\text{-}a)$ from the semi-infinite wake side vortex sheets, strength $2aT$. For any pass with mirror symmetry about $\theta = \pi/2$ there is no net flux and so no power from the bound vorticity. Wilson [5,11] showed conservation of vorticity means the sums of the |Γ| 's is $2ma(1\text{-}\, a)\, T$ so that for **any** (at least left-right symmetrical) shape circuit of windward and lee passes with **any** two circulations the total $C_P = 4a\,(1\text{-}a)^2$. For equal passes Wilson found the rotor crosswind lift coefficient to be $-\pi m\, a^2(1\text{-}\, a)$ but no crosswind wake deflection so the bound vorticity produced a net pressure force on the side of the rotortube.

Returning to fixed pitch and the BEM method, the above construction of a justification for Wilson and Lissamen ignoring the lateral induction contradicts considering the leeward pass to be in the wake of the windward, yet the DMST method does both.

5. VAWT LEEWARD FLOW AND ROBUST PITCH CYCLE

As the $\theta = \pm\frac{1}{2}\pi$ peak power points are a full diameter apart in the direction of the wind, it is more reasonable than common induced windwise flow to approximate the windward wake as fully developed at the leeward pass. Indeed this DMST element is the converse of the above treatment of the windward pass as being too upstream to be affected by the leeward wake.

In the 3D spherical high m limit of symmetry about the wind axis, then the downstream oncoming flow to the annulus at downstream normal $\underline{\mathbf{r}}_l$ at angle $\pi/2 - \phi$ to $\underline{\mathbf{T}}$, is $\underline{\mathbf{T}} - 2\, a_0\, \underline{\mathbf{r}}_w$ and the leading new induced flow is radial $a_0\, \underline{\mathbf{r}}_l$, so that the normal velocity leads as

$$\underline{\mathbf{W}} \bullet \underline{\mathbf{r}} = \underline{\mathbf{T}} \bullet \underline{\mathbf{r}}_l - 2\, a_0\, \underline{\mathbf{r}}_w \bullet \underline{\mathbf{r}}_l - T a_0 = T(\sin\phi - 2a_0 \cos(\theta+\phi) - a_0) \qquad (15)$$

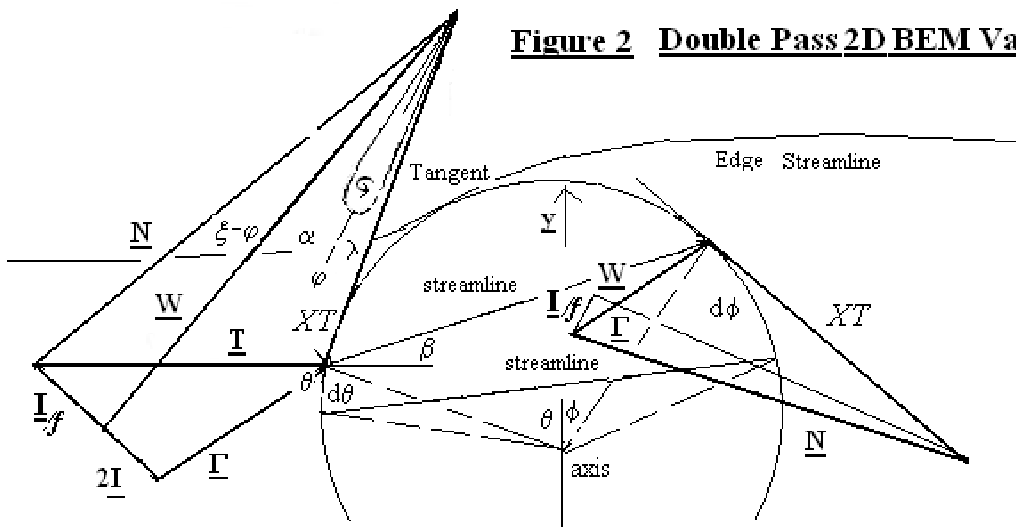

Figure 2 **Double Pass 2D BEM Vawt**

Again because at the equatorial plane this represents the 2D situation, these terms will also lead there. To get all the 2D terms, let the average velocity drop at the leeward pass have components $2d$ and $-2n$ with induced flows at the blades d/f and n/f, respectively. Then the lee oncoming stream is

$$\underline{\mathbf{D}}/T = (1-2u-d/f)\underline{\mathbf{x}} + (2v - n/f)\underline{\mathbf{y}} \qquad (16)$$

As in Fig. 2 define a leeward $\phi = -\theta$, and note the reversed leeward circulation now gives $\underline{\mathbf{z}}$ into the page. As before, (2) $\underline{\mathbf{I}} = E(\underline{\mathbf{z}}\, \mathbf{x}\, \underline{\mathbf{W}})$ generates the simultaneous linear

$$d = a_0 \sin\phi + En/f - 2hv \qquad n = a_0 \cos\phi - Ed/f + E - 2Eu \qquad (17)$$

Cross substituting these gives solutions analogous to the windward ones, but with the extra terms due to the $2u$ & $2v$ disturbances in the oncoming flow

$$Hd = a_0\sin\phi + Ea_0\cos\phi/f + E^2/f - 2E^2u/f - 2Ev \qquad Hn = a_0\cos\phi + E - Ea_0\sin\phi/f - 2Eu + 2E^2v/f \qquad (18)$$

Using (7) and taking a_0 small gives a net rotor lift coefficient by (1) of about $-2\pi m a_0^2$. The average $a = 4\, a_0/\pi$ so this is $\pi^2/8$ Wilson's side force. Next forming the component $n\sin\phi - d\cos\phi$ of the leeward velocity change reacting to the leeward thrust, and then dividing out the E factor after leading term cancellation, gives by (3)

$$H\ \underline{\mathbf{W}}\bullet\underline{\mathrm{r}}/T = \sin\phi - a_0/f - E\cos\phi/f + 2\{u\,(E\cos\phi - \sin\phi) + v\,(E\sin\phi + \cos\phi)\} \qquad (19)$$

Substituting for the windward Vawt u and v from (7) gives

$$H\ \underline{\mathbf{W}}\bullet\underline{\mathrm{r}}/T = \sin\phi - a_0/f - E\cos\phi/f + 2\, a_0 \cos\ \theta+\phi\ + 2E\cos\phi \qquad (20)$$

At $\underline{\mathbf{I}}=\underline{\mathbf{J}}$ the ϕ average d is less than the θ average u by $2E^2 = 2m^2a_0^2$, and the mean n is less than mean v by about $4Ea_0/\pi$. So the mean leeward I would be less than the windward and so the mean leeward $W' = I/E$ is less than the windward $W' = I/E$ all by a fraction $O(E)$.

The Hawt has a definite, if implicit, radial $\underline{\mathbf{J}}$-$\underline{\mathbf{I}}$ component expanding the streamtube as the wind is slowed axially. The corresponding vertical component will be assumed nil for a 2D Vawt, and mass conservation applied in the horizontal plane. Equating (8) x dθ to (20) x dϕ, allows in principle solving θ in terms of ϕ. The f intensification factors at the blades should not apply in this balance of average flows.

$$(\sin\theta - a_0 - E\cos\theta)\, d\theta \approx d\phi\,\{\sin\phi - a_0 + E\cos\phi + 2a_0\cos\ \theta+\phi\} \qquad (21)$$

Note that the two normal velocities in brackets identically vanish at $a_0 = \sin\theta = -\sin\phi$ as $m \downarrow 0$ which is then a point of tangency. Chaney et al[10] had the equivalent of (4) and this equation with an implicit representation of its *{... }* bracket but evaluated it indirectly by considering the streamtube energy fluxes from far upstream to far downstream . They assumed the near wake flow to be everywhere windwise whereas that is only true on average (here) and the Hawt near wake [6] has crossflow which is dissipated at a power loss, so their theory is still biased (like the DMST's no crossflow anywhere). In the high X limit this gave $\{ \sin\phi + a_0(1-(\sin\theta + \sin\phi)^2 \}$ which is more than the above by $a_0(\cos\phi - \cos\theta)^2 = O(a_0^3)$. Spherically (21) has the same leading terms in the brackets but $\cos\theta\, d\theta$ and $\cos\phi\, d\phi$ for the area segments ds . In 2D to first order in a_0, expanding θ about ϕ gives

$$d\{\sin\phi(\theta-\phi)\} \approx d\phi\,(2a_0\cos 2\phi + 2E\cos\phi) = d\{a_0\sin 2\phi + 2E\sin\phi\} \quad (22)$$

$(\theta-\phi)/2$ is the mean deviation of a streamline inside the rotor and can be expected to vary from $v=E$ at $\theta=0$ windward to $2v-n=E$ at $=0$ leeward so the average is indeed E so

$$\theta-\phi = 2a_0\cos\phi + 2E + O(a_0^2) \qquad (23)$$

confirming for $m\to 0$ that the side point where $\sin\theta = a_0 = -\sin\phi$ is the tangent streamline of zero net flux. Anticipating a_0= .174 gives the tangency at about 10 ° versus about average 11 ° in the Wilson model [7]. Spherically (20) becomes

$$(\theta-\phi)\sin 2\phi \approx 4(3\sin\phi - 2\sin^3\phi - 1)/3 \quad (24)$$

which does not confirm the side points. But perversely at such finite a_0 integration of (21) for 2D finds the zero of lee normal velocity at the side ϕ is reached before the side θ [10], but not much spherically, due to the greater area differential with the cosine weighting in ds. A heuristic resolution would be to average this θ at $\phi = -a_0$ with the above $\theta = a_0$ as the end of the windward pass. Chaney et al instead zeroed the power in the entire gap, postulating a sideband of recirculating or choked flow[10]. But there is no experimental evidence of this and it does not appear in the exact Wilson solution which has even stronger axial interference on the sides than the tangent blade case. Perhaps that solution should be iterated numerically for its expanding wake.

But a more mundane explanation is that at finite a_0 the more strongly diverging 2D interpass flow implies pressure gradients and net non-neglible pressure forces on the streamtube momentum balances in both pass BEM's, reminiscent of the role of pressure in Wilson's 2D side force.

So proceeding under this caution, expanding (15) to $O(a_0^3)$ gives

$HW \bullet r/T \approx \sin\phi - a_0/f + (2-1/f)E\cos\phi + 2a_0\cos 2\phi - 4a_0\cos\phi\{a_0\sin 2\phi + 2E\sin\phi\}$ (25)

Since the fractional difference in W's was found to be $O(E)$, this $W\sin\gamma$ smaller by only $O(a_0^2)$ than the windward means leeward γ's become bigger as $X\downarrow$ (and $a_0\downarrow$ with it at fixed σ), as is clear must be so statically at X=0. Since the continuation of θ to leeward is - ϕ and the sign of φ is reversed in the blade frame, to leeward

$$X\varphi \rightarrow \sin\theta + a_0/f - 2a_0\cos 2\theta + 8a_0^2\cos^2\theta\sin\theta + O(ma_0...a_0^3) \qquad (26)$$

The 90 ° range averages of | φ| is unchanged from the windward (9) for small a_0, so their average f's can be approximated as the same. The extra $2a_0\cos 2\phi$ terms making the leeward φ curve much flatter than the windward, as seen in Fig 3 for a_0=.172 & f=1 Dashed in are and the Fourier

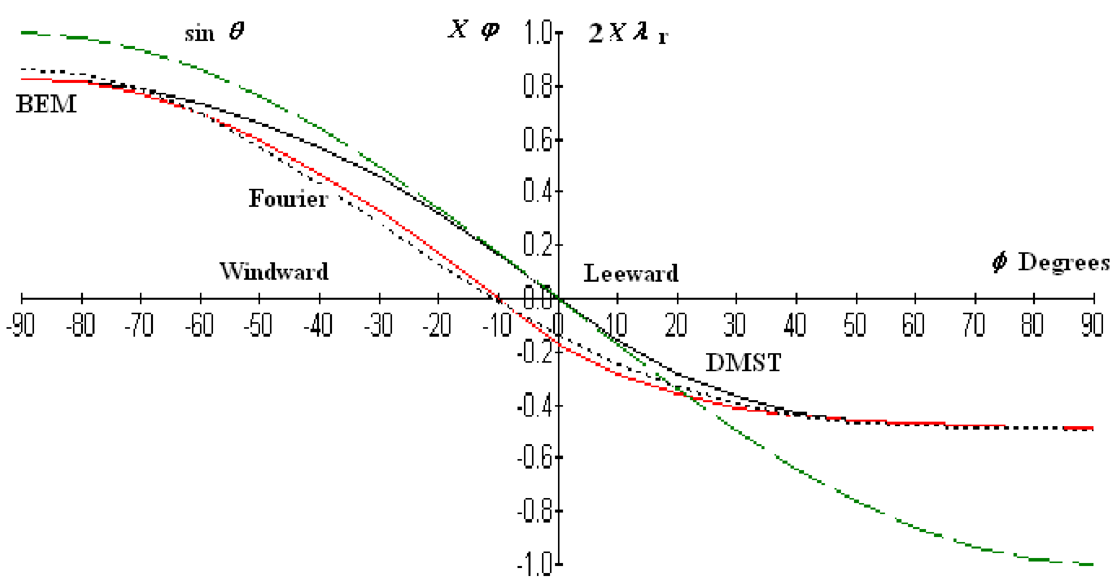

Fig 3 Apparent Wind Angles φ @A=.35

approximation for $X\varphi$ of .025+.68 sin θ-.16 cos 2θ and $X\varphi$ = sin θ at a_0=0 (and 1-a that for the ideal). The mean arises from the non zero crossover and especially the final a_0^2 term in (26). Half this φ is the pitch cycle λ_r for a robust benign peak and such a mean ¾ chord noseout has indeed been shown by measurements Fig. 8.15 [5] to give a broader operating Vawt peak. The second harmonic, roughly ½ the leeward-only $2a_0$, makes this cam cycle significantly different from the standard sin θ from nose pushrods offcenter upwind, making the windward greater and more peaked, the leeward smaller and flatter and the windward zone smaller than the leeward.

6. NET BEM VAWT POWER

From the spherical (25)

$$C_{P3}=8a_0 \int_0^{\pi/2} \cos\phi \, d\phi \{\sin\phi - a_0/f + 2a_0\cos 2\phi - 8a_0^2(3\sin\phi - 2\sin^3\phi - 1)/3\}^2 \quad (27)$$

Ignoring terms above cubic in a_0 gives the total C_{P3} as $16a_0^2\,(1/3- a_0+ .3a_0^2)$ with an optimum of .47 split .27/.20 at $A=2a_0=.362$ vs Chaney's numerical .49 at $A=.4$ because of higher order terms including his higher lee normal velocity (and no significant 3D 'choking'). More importantly his results show a significant divergence with DMST with increasing interference without 'choking' but he did not dissect this first, as will now be done here in the easier 2D. In 2D ignoring terms above cubic in a_0 gives

$$H^2 dP_2/ ds /2\rho T^3 a_0 =(\sin\phi - a_0/f +(2-1/f)E\cos\phi+2a_0\cos 2\phi)^2-4a_0\sin 2\phi\{a_0\sin 2\phi+E\sin\phi\} \quad (28)$$

Integrating over ϕ from 0 to π by the average values of sinusoids with no average effect of the E skew in the center streamline, nor $2E$ and cancellation of the average a_0^2 terms gives

$$\text{lee } H^2\, C_P = \pi f\, q_0 \{1-8q_0/\pi +2q_0^2 +(2-1/f)^2E^2-16a_0/3\pi\}+O(a_0^4) \quad (29)$$

Where again f becomes a mean. Note the (windward pass) endpoint reduction was of the same $O(-a_0^4)$, and the net effect should be negative as the lee power rate will prove much less than the windward .The final term in (23) is a negative effect of the leeward pass being in the wake of the leeward and will be conservatively approximated as $16q_0/3\pi$. So combining with (12) and expressing in terms of $A=2\,q_0$ and f now an average over both passes

$$\text{Total } (1+1/4\, m^2A^2/f^2)^2\, C_P \approx \pi f A\{1-16A/3\pi+(\tfrac{1}{2}+1/4\, m^2)A^2\}+O(-A^4) \quad (30)$$

peaking at A=.348 giving C_P /f=.338 windward +.176 lee for a total of .514 vs. the Betz limit 16/27 for the optimal pitch or a Hawt[2]. In the worst case choked interpretation of the 2D imbalance the C_P /f peak was lowered to .48 split .30/.18 at best interference A=.3 [10]

The lower leeward power is inevitable because the power depends on the integral of the square of the normal velocity (4) whereas the flux integrals of the first powers must be equal and are so by the extended leeward range making up for the lower peak leeward normal velocity. The 1.92 split ratio compares reasonably with 1.72 taking the circulations in the ratio of the peak apparent wind angles for the Wilson exact solution.

In (24) {..}'s linear term in A which includes the interpass term is exactly the same as in the common interference model (11) and is generally indifferent as to how the induction is shared between the two passes. Yet the quadratic coefficient is ½ vs. Lissamen's ¾ which explains the lowering of the optimum A from .401 to .348 with a narrower peak in A and more breathing room from reversed wake flow.

Ignoring lateral induced flow and just solving the windwise component of (2) gives $u= a_0\sin\theta$ and $d=a_0\sin\phi$, which are not even the windwise components (7) and (14) of the real induced flow at finite m. Pawsey [7] took the lee pass net induced flow as $3a_0\sin\phi$ [7] so the above limiting quadratic coefficient comes out to 15/16 for an extremely broad peak C_P/f of .605 (close to McCoy's .617 [13]) split .49/.11 at the maximum allowable A=.5 . This reflects an inherent positive effect in double pass models; that due to the flux squared term in (4) there is a net gain in quadratic coefficient by unequally splitting the common induced flow into two passes. Strengthening the downwind axial induced flow to the expanding DMST $2a_0\sin\theta+a_0\sin\phi$ is difficult to solve analytically but numerical integrating outwards from the centerline to stop the lee at ϕ=0 at non-zero θ but optimistically continue the unbalanced windward until θ=0 gives a broadly peaked C_P/f of .57 split .46/.11 around A=.43. With variable pitch for constant angle of attack and axial induced flow, the ultimate DMST C_P/f is 16/25 like one ideal actuator in the wake of another [5, 13], but this is biased relative to Wilson's full vortex analysis of this ideal situation [11]. Here including the lateral flow reduces the fixed pitch power by about 11% and narrows its peak and redistributes it from the windward being four times more powerful to less than twice.

Fig 3 also plots the apparent wind angle for Pawsey's DMST at the same A showing how different its angle of attack variation is from the BEM. There is no interference on the sides (under- induction not the BEM lateral over-induction versus the ideal uniform axial) and no shift to windward of the φ crossover point . The DMST axial flow methods all underestimate the reduction of windward φ by a given a_0 by $\sin^2\theta$ so by ½ at π/4 and thus the sharpness of the angle of attack cyclic peak, and the lowering of the blade's net thrust by wider excursions from the thrust peak.(Appendix 2). The neglected crosswind lateral reaction induced/ bound circulation flow at finite X distorts the actual side angles of attack further. These basic inaccuracies undermine treatment of dynamic stall in the DMST codes [5].

The bracket { } in (24) has the value .47 at the peak at m=0, so the **I**=**J** $O(m^2A^2)$ corrections on the two sides again virtually cancel, as exactly for constant Γ. This can be attributed to little kinetic energy loss from torque reactive crosswind flows as the windward and leeward passes cancel .

6. THE OPERATING POWER PEAK IN VAWTS AND HAWTS

Whilst the above Vawt peak power deficit versus an ideal Hawt (or variable pitch Vawt [2,11]) of 10% is modest enough, the drag penalty is higher for the Vawt and its tangent or passively articulated blades have A varying inexorably with X and so a narrow operating power peak, even if stall is ignored.

The standard and simplest electric generator is synchronously tied to the grid which stiffly loads the windmill to run at virtually constant angular velocity Ω in fixed proportion to the grid frequency. Then it is vital that the wind turbine have a broad C_P, as X changes inversely with wind speed T [2].

For the Hawt, the constant e 'tangent' power drops off-optimum as $(\delta A)^2$, so as δX^2 for small variations δ away from the design values, whereas for a robust Hawt the drop is as δX^4 [2,12] Robust Hawt blades fixed at halfway between the design apparent wind and the tangent keep their off-peak induction below optimal, for low blade loadings and good wind park efficiency [2]. Constant chord tangent Hawt blades need high design X with high drag loss not to be stalled, and lose induction as X falls to stall quickly, whilst robust optimal fixed Hawt blades have higher inductions for smaller $\delta\alpha$, as numerically illustrated in [2]. At X 50% above design, tangent Hawts[2] overinduce causing reversed wake flow and high, fluctuating aerodynamic loads that cause high noise, vibration and stress for low power, especially of arrays[2]. The high induced flow soon lowers φ and α below the threshold for positive thrust difference between leading edge suction and drag, whilst the large normal force remains completely unproductive. Thus there are narrow operating ranges in X between underinduced stall and overinduced reverse flow and in α between drag and stall zeroes of thrust.

The Hawt and Wilson exact Vawt $C_P = 4A(1-A)^2$ have second derivative in A of -8 at the optimum whereas the BEM 2D & spherical Vawt's have -7.4 and -6.7 respectively, so the tangent Hawt and Vawt peaks have about the same small breadths in A and so X (whereas DMST peaks are much broader). But changing from horizontal to vertical axis exacerbates the tangent design drag by $\pi/2$ and likewise narrows the positive mean thrust range of α amplitude over static especially with the peaky variation of α found here. The Vawt over-induced and choked zones need study if only for vibration, yet even tunnel flow visualisation seems lacking.

Vibration is a major Vawt constraint with its oscillation of large blade lifts for weak thrust. Pairing blades has much more of a structural benefit for the Hawt with less vibration penalty but has been risked more often in Vawt's. Avoiding the narrow power peak and high X reversed turbulent flow of fixed speed operation with more expensive variable speed generation makes the Vawt pass through a wide range of frequency Ω and its harmonics and risks resonances and fatiguing. An underestimated resonance forced the very large Cap Chat Darrieus to be run synchronously[5].

A Hawt blade can be cambered for higher α_{stall} and maximum lift coefficient than a double-acting Vawt blade . Even (cyclic) cambering of a tangent chord blade would not increase its leading edge suction thrust whereas all pitched windmill blades can exploit the much larger and less critical normal force for much larger thrust coefficients than the tangent blade. The centrifugal secondary flow [14] in a Hawt blade can increase the stall angle whereas the net thrust in tangent Vawt-like pure pitching shows only dynamic mitigation of deep stall [15].

The tangent blade Vawt can be converted to a broad robust peak and benign off-peak [2] by articulating the blade to follow the fixed $\lambda(\theta)=\varphi_d/2$ cycle of (26) which also lowers the design drag. Whereas, constant e=½ passive pitch still has reversed flow at high X, and a narrow peak in X, (though less curtailed by stall, at low X). Sharp [9] puts the center of mass ahead as well as outboard of the blade pivot to distort e as X rises towards the lee pass than the windward, which will palliate the high X over-induction problem of passive pitch, as should nose-out of fixed blades. Centrifuging a mass against a blade cam with quadratic torque characteristic as λ^2 would make α passively diminish asymptotically as $(\varphi-\alpha)^2$ and so as φ^2, and so e as m'. McDonnell Aircraft [16] calculated optimal α (X) square wave in θ that also gave $e(\theta)$ decreasing ultimately as $1/X$, but tragically did not test it or much else with their computer-articulated Giromill.

But Hawt blades can also pitch with very good cantilever bearings at their hub or effectively alter their outer span pitch with flaps, and need only do so on the time scale of T not $1/\Omega$. However they cannot alter their twist, so an ideal pitch-articulated Vawt [2]can be optimal over a wider speed ratio range than a variable pitch Hawt, slightly reversing the overwhelming advantage the Hawt has with fixed blades.

8.CONCLUSIONS

Wilson and Lissaman's common induced flow Vawt model has been rationalised contrary to the DMST's increase of the peak power and interference. DMST separation of the passes contradicts the cancellation of lateral velocities Wilson assumed. The BEM includes them which drops and narrows the double pass power and best interference below the Wilson values. It predicts the points of net flow tangency dividing the two passes to be correctly upwind of the crosswind diameter. Whereas the DMST fails to and so miscalculates the upwind angle of attack sequence important in treating dynamic stall

The Vawt operating power peak is inherently narrower than the (robust) Hawt's. At 1.44 the design X_d the fixed or passive pitch Vawt begins reversed wake flow which invalidates the leeward pass model. The actual and possible preliminary choked flow need to be observed and compared with the Hawt reversed flow which has been better observed even though Hawts can avoid this condition with minimal design pitch. Vawts can less easily avoid it with quadratic cam centrifugal pitch articulation or the robust fixed pitch cycle, which also has half the intrinsic drag penalty. However the existing centrifugal passive pitch schemes do not avoid flow reversal at high X. Like so much Vawt design they have suffered (fatally) from a general lack of impartial, Hawt-correlated analytical insight, which if supplied at all by this paper will have fulfilled its purpose.

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APPENDIX 1: CORRECTIONS

UNSTEADY AERODYNAMICS - In unsteady flow theory, oscillation of the ¾ chord angle of attack α about an apparent wind W, reduces α by the complex linear operator $C(k) = F\text{-}iG$ the Theodorsen function where the k are the harmonic multiples of $\Omega c/2W \lesssim c/2r$. The tangent Vawt interpretation would be the blade lift is normal to apparent wind at the corrected (lift) angle $|C|\varphi(\theta - \Delta\theta)$ where for the first harmonic the lag $\sin \Delta\theta = G/|C|$. Then (4) is just $dP/ds = 2\rho T a_0 |C|^2 W^2 (\varphi(\theta - \Delta\theta))^2$ but φ itself is harder to solve. With q_0 scaling as $a_0 |C|$ the optimum power would scale as $|C|$ or about 1-.71k and the drag loss as 1/ $|C|$ from the bigger a_0.

Hein's Fig 7.6 [16] verifies a reduction of the mean thrust at k=.08 and .1 for sinusoidal pitch of a NACA 0012 at Re=3x10^5. At an optimum k=.055 Hien shows leading edge suction overcomes drag at 4.5° 0-peak amplitude, and the mean thrust coefficient peaks near the k=0 peak of .06 at 12° amplitude. Then it falls to .04 at 21° but dynamically avoids the k=0 zero at 24° and then rises above .06 above 28° amplitude . The persistence of a (drag) threshold at the (static) stall angle denies decomposition at large angles of the significant harmonic mixture of φ in Fig 3.

DRAG: Generally one can correct the above ideal fluid power for real fluid profile drag $\frac{1}{2}\rho W^2 B c C_D$. This rises at high speed ratios to decrease C_P by $2\varpi(W/T)^2 X C_D \approx 2\varpi X^3 C_D = -X^2 C_D A/e$ Thus drag can be viewed as a X^2 increase in the negative linear coefficient in the A cubic (24), which favours lower A's, i.e. lower solidity at a given X. This term is about π less for a tangent Hawt segment where the swept width is $2\pi r$ vs. $2r$, but the single pass requires twice the a_0 for the same A; whereas the Vawt always has the extra parasitic drag of any blade support arms and guys. In the Hawt the optimum X is a tradeoff between the drag penalty increasing as X^3 and the ideal lift power [12] rising with X, as the torque reactive swirl energy lost in the wake decreases. Here the two opposite passes leave little reactive crossflow energy in the wake [11], so the rise with X is much sooner due to unstalling. Designing the peak angle of attack as the static stall value of say 12° gives X_d=3.9e . There δC_P is .11e for an average C_D of .02 up to stall, so at design the robust cyclic should outperform the tangent by about 10% due to half the drag penalty. Then the design $\varpi = Bc/4r$ is .045/e^2 which gives Hien's optimal $k=c/2r$ =.055 for B=2.5/e^2

BLADE ANGLING & ARCING- If there is no wind the flow is purely the curved self wind. Just as for a cambered blade the zero lift setting is when the blade midline at the ¾ chord point is aligned with the flow. Then the common blade setting of midchord tangent can be considered a built-in nose-in angle of attack of $c/4r$ windward for e>1 and $-c/4r$ leeward for e<1. Experiments [5,15] have shown that ¾ chord tangency is in fact close to optimal for a symmetrical blade Vawt and definitely superior to the common naive tangency at midchord.

The dominance of tangential self-wind at high X has suggested to many curving tangential finite chord Vawt blades to match the curvature of their path to reduce their ultimate form drag, though still air quantification is lacking. Then the only normal velocity component towards the blade surface is from the real wind. As above the zero lift reference for arced blades is the tangent at the ¾ point so that is again the evaluation point for φ and the reference point defining the blade θ. The camber would extend the angle of attack range on the lee pass vs the windward which is the opposite of that needed at stall threshold in this high X theory, and extra lift due to camber does not have leading edge suction for a thrust benefit. But it might help powering up through low X.

TIP EFFECTS : The vertical axial expansion flow neglected in 2D reduces the downwind pass induced flow, to reduce choking and improve performance a bit. Towards the tip keeping A constantly optimum implies the chord must vary as $1/f$ which resembles a elliptical rounding of the tip. The effective loss of span is 1.4 $R\sin\varphi/Bb$.

Vawt proponents generally used the aspect ratio induced drag correction of single wings. The minimum and uniform self-induced velocity of an elliptic blade semispan b due to a plane sheet of vortices trailing to infinity is $L/ 2\pi\rho Wb^2$. The LHS of (1) gives $L=2\rho\ 2b\ 2\pi R/B\ W \sin\varphi$ I, also perpendicular to **W**. So a minimum for the ratio of the blade induced velocity to the continuous actuator's is $2R\sin\varphi/Bb$, so again keeping the total at the blade optimal vs. T, the power is reduced by this fraction, at least 40% greater than the Prandtl loss.

Certainly to avoid the low lift and high drag of stall for starting one wants low aspect ratio $2b/c$ and an elliptical planform. Crudely the Prandtl tip half-"ellipse" is about 5 times the loss of span long so its ideal planform is elliptical when the average $6Rm(2/\pi - a_0)/Bb \approx 1$. So the aspect ratio $2b/c$ of this 'universal planform' is

$3.5e(2/\pi - a_0)/a_0$ or $9.3e$ at the design optimum a_0 of .172 (with about 20% tip loss), promising low for avoiding stall and self-starting for cyclic pitch $e_d = ½$ with its smaller angles of attack in the first place.

WIND SHEAR- Robust Hawt's [2] only lose optimality to $O(\kappa^4)$ due to the gradient $\kappa T/r$ in the true wind T, and with three blades their rotor moments are smooth to $O(\kappa^2)$. Considering constant chord Vawt spanwise effects, whilst a_0 is independent of R, it does decrease vertically with the wind gradient losing optimality as $O(\kappa^2)$. So ideally the Vawt chord c should increase with T vertically.

More important are the gradients in wind cubed power $3\kappa T^3/R$ and the speed ratio $-\kappa X/R$. Upright Vawt blades have lowest X and highest dP/ds at their tops in the strongest wind. Therefore stall first appears dramatically by dropping the heavy windward blade loadings at the top of the straight blade Vawt, promoting its super vibration. Inclining straight Vawt blades outwards by $\Phi = \kappa$ for uniform X seems a minimum precaution. 3D curved blade Vawts are always stalled at the blade ends and stall spreads progressively as X is reduced so they can be run at widely variable X such as synchronously.

Inclining straight blades also matches the capture area better to the windpower shear. In the common induced flow model [3], inclination does not change the <u>average</u> induced flow in the wind direction nor the maximum power per unit vertical distance *ie* C_p. (However again the drag correction is as the inclined length.) For a straight blade of a set length, the inclination Φ (top out) for the most swept power at a given mean radius R is $\Phi = \kappa b^2/R^2$ and inclining by $\Phi = \kappa$ for constant X gains C_P over upright by the fraction $\kappa^2(b^2/R^2 - ½)$.

Of course Vawt's capture twisted winds better than big Hawt's. Those even lose power as their power-consuming yaw motors act slowly in response to wind shifts. The cube of the cosine of the wind yaw (10) overestimates the loss a bit, because by pressure effects the flow induced by the yawed wake is only **<u>I</u>** on average[17], and again in (4) this raises the power.

APPENDIX 2 NOTATION

A the net interference factor $p_0 + q_0$ at the blade for both passes

a_r $=1-\varphi/\xi \rightarrow I/T\sin\theta$ the local (radial) interference factor

$a_0 = Xg = XI/W = Xel\varpi = Xel\ Bc/4R = BKe\ l$ representative of the non-dimensional actuator induction I/T at high X

B the number of blades

b blade semi-span

b_0 induction on the leeward pass for leeward e

c the local blade chord

C_D sectional drag coefficient

C_L sectional lift coefficient

C_P - rotor coefficient of power / undisturbed kinetic energy flux through the swept area

d upwind component of leeward $\underline{\mathbf{I}}$/T

f Prandtl tip correction factor

$C(k)$ the complex Theodorsen function

$e = \sin\alpha / \sin\varphi$ effective factor on blade chord

$H = 1 + g^2/f^2 = N^2/W^2$

$E = e\varpi = I/W$ ratio of the half the velocity change to the apparent windspeed W windward

F real part of $C(k)$

G imaginary part of $C(k)$

$K = \Omega c/2T$ reduced frequency based on semichord & true wind

$k = \Omega c/2W \rightarrow c/2R$ reduced frequency based on semichord & apparent wind

l lift slope correction factor for thickness (and unsteadiness)

m inverse speed ratio $T/\Omega r = 1/X$

n crosswind (-$\underline{j}$ unit vector) component of $\underline{\mathbf{K}}$/T

P the dimensional power per unit length of span

$p_0 = b_0/f$

$q_0 = a_0/f$ the total induced flow at the blades

$t = 2\pi \sin\varphi \, r/B$ spacing between tip vortex sheets

r (spherical) radius

R (local) radius to axis of rotation

T windspeed

u upwind component of $\underline{\mathbf{I}}/T$

v crosswind ($\underline{y}$ unit vector) component of $\underline{\mathbf{I}}/T$

X local speed ratio $\Omega R/T$

W speed of the apparent wind

$\underline{r}$ unit radial vector normal to the actuator surface

$\underline{x}$ unit vector in the downwind direction

$\underline{z}$ unit vector along the blade span

$\underline{\mathbf{I}}$ the induction, half the velocity change the blades produce downstream in their wake

$\underline{\mathbf{J}}$ the actual induced flow velocity vector at the rotor

$\underline{\mathbf{\Gamma}}$ the wake velocity $\underline{\mathbf{T}}$-2$\underline{\mathbf{I}}$

$\underline{\mathbf{D}}=\underline{\mathbf{T}}+\underline{\mathbf{J}}$ the net airflow at the rotor relative to the ground

$\underline{\mathbf{L}}$ airfoil lift vector

$\underline{\mathbf{N}}$ the no-lift or nominal apparent wind $\underline{\mathbf{T}}$-$\underline{\mathbf{S}}$

$\underline{\mathbf{T}}$ True undisturbed wind vector

$\underline{\mathbf{W}}$ The net apparent wind vector $\underline{\mathbf{T}}$-$\underline{\mathbf{S}}$ +$\underline{\mathbf{J}}$

α - ¾ chord angle of attack to $\underline{\mathbf{W}}$

α_{stall} the stall onset value of α

β mean deviation of streamline inside rotor

ϕ - Vawt downstream azimuthal angle (θ without streamtube expansion)

φ - The true or complete apparent wind $\underline{\mathbf{W}}$ angle to the blade path (windward)

Γ – bound vorticity density / circulation

λ - blade pitch or angle of the 3/4 blade chord to the blade path

κ non-dimensional vertical gradient in the wind

ρ fluid density

σ - true local solidity ==$Bc/2\pi R$ blade chords/circumference of blade travel

θ- windward angle between $\underline{\mathbf{T}}$ and $\underline{\mathbf{r}}$

$\underline{\boldsymbol{\theta}}$ - unit vector in azimuthal direction of increasing θ

$\underline{\boldsymbol{\phi}}$- unit vector in azimuthal direction of increasing ϕ

ϕ - downwind azimuth angle

Φ - small inclination of blade span to vertical

ξ - The nominal or No-lift apparent wind $\underline{\mathbf{N}}$ angle to the blade path

$\varpi = \pi\sigma/2 = Bc/4r$ half the net blade chord divided by the diameter

Ω- angular velocity Ω of rotation in radians per unit time

subscript $_d$ denotes design value

$_w$ denotes windward

$_l$ leeward

$_2$ 2 dimensional

3 3 dimensional (spherical)

$\underline{j}$ implies the average of j over θ

prefix δ denotes a small variation of

{..} denotes the curly bracket term in an equation